Final Exam

May, 13$^{\text{th}}$ 2024

- Time Limit: 10:30 am - 12:30 pm.
- **No** books, course notes nor electronic devices are allowed.
- The problems are on the other side of the paper.
- Upon finished, the examination paper has to be submitted together with your answer book.

Course check (40%)

1. Define what does it mean that two norms are equivalent. What happens in finite dimensional vector spaces like $\mathcal{M}_n$?

2. Give the definition of matrix norm and spectral radius, give their relation and prove it.

3. Given a matrix A with all entries strictly positive, what can be deduced on the eigenvalues of biggest modulus and on their associated eigenspace? (Put together the 3 Perron Frobenius results).

4. Given $A, B \in \mathcal{H}_n$, define the notation $A \succeq B$ and prove:

   - Given any invertible matrix[1] $P \in \mathcal{M}_{p,n}$: $P^*AP \succeq B \implies A \succeq P^{-*}BP^{-1}$.
   - $A \succeq B \succ 0 \iff 0 \prec A^{-1} \preceq B^{-1}$.

5. Provide the definition of the tensor rank.

Problem 1 (25%): Normal matrices.

In this problem we will work with so called "normal matrices" that are matrices $A \in \mathcal{M}_n(\mathbb{C})$ satisfying:

$$A^*A = AA^*.$$

Let $A = [a_{ij}] \in M_n$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly equal, we do not assume here that A is diagonalizable). We are going to show that the following statements are equivalent:

(a) A is normal.

(b) A is unitarily diagonalizable (i.e. there exists U unitary such that U^*AU is diagonal).

(c) $\sum_{i,j=1}^n |a_{ij}|^2 = \sum_{i=1}^n |\lambda_i|^2$.

(d) A has n orthonormal eigenvectors.

Answer the following questions:

1. Show that *(b)* $\implies$ *(c)*.

   Correction: *If there is a unitary V such that $A = V\Lambda V^*$ and $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$, then:*

   $$\sum_{i,j=1}^n |a_{ij}|^2 = tr(A^*A) = tr(\Lambda^*\Lambda) = \sum_{i=1}^n |\lambda_i|^2.$$

   □

2. Show that any diagonal matrix is normal. Show that any matrix unitarily similar to a normal matrix is also normal.

3. Show that *(d)* $\implies$ *(a)* **Correction:** *Let us denote $u_1, \ldots, u_n$, the n orthonormal eigenvectors of A respectively associated to the eigenvalues $\lambda_1, \ldots, \lambda_n$. The unitary matrix $U = (u_1, \ldots, u_n)$ and the diagonal matrix $\Lambda = Diag(\lambda_1, \ldots, \lambda_n)$ then satisfies $A = U^*\Lambda U$. One can then conclude with Item 2.* □

[1] Recall that $P^{-*} = (P^*)^*$.

4. Let $A \in M_n$ be partitioned as
$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$
in which A_{11} and A_{22} are square. Show that A is normal if and only if A_{11} and A_{22} are normal and $A_{12} = 0$.
Correction: *If A_{11} and A_{22} are normal and $A_{12} = 0$, then $A = \left(\begin{smallmatrix} A_{11} & 0 \\ 0 & A_{22} \end{smallmatrix}\right)$ and of course:*
$$AA^* = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} A_{11}^* & 0 \\ 0 & A_{22}^* \end{pmatrix} = \begin{pmatrix} A_{11}A_{11}^* & 0 \\ 0 & A_{22}A_{22}^* \end{pmatrix} = \begin{pmatrix} A_{11}^*A_{11} & 0 \\ 0 & A_{22}^*A_{22} \end{pmatrix} = A^*A. \quad (1)$$
Conversely, if A is normal, then
$$\begin{pmatrix} A_{11}A_{11}^* + A_{12}A_{12}^* & * \\ * & * \end{pmatrix} = AA^* = A^*A = \begin{pmatrix} A_{11}^*A_{11} & * \\ * & * \end{pmatrix}$$
so $A_{11}^*A_{11} = A_{11}A_{11}^* + A_{12}A_{12}^*$ which implies that
$$tr(A_{11}^*A_{11}) = tr(A_{11}A_{11}^* + A_{12}A_{12}^*) = tr(A_{11}A_{11}^*) + tr(A_{12}A_{12}^*) = tr(A_{11}^*A_{11}) + tr(A_{12}A_{12}^*)$$
and hence $tr(A_{12}A_{12}^*) = 0$. Since $tr(A_{12}A_{12}^*)$ is the sum of squares of the absolute values of the entries of A_{12}, it follows that $A_{12} = 0$. Then $A = \left(\begin{smallmatrix} A_{11} & 0 \\ 0 & A_{22} \end{smallmatrix}\right)$ and we saw in (1) that in that case A is normal if and only if A_{11} and A_{22} are normal. □

5. Show that *(a)* $\implies$ *(b)*. **Correction:** *Consider the Schur triangularization $A = UTU^*$, in which $U = [u_1 \dots u_n]$ is unitary and $T = [t_{ij}]$ is upper triangular. If A is normal, then so is T (see Item 2). The preceding results ensures that T is actually a diagonal matrix, so A is unitarily diagonalizable.* □

6. Show that *(c)* $\implies$ *(d)* **Correction:** *With the same Schur triangularization $A = UTU^*$ as in the previous solution, the diagonal entries of T are $\lambda_1, \dots, \lambda_n$ in some order, and hence $tr(A^*A) = tr(T^*T) = \sum_{i=1}^n |\lambda_i|^2 + \sum_{i<j} |t_{ij}|^2$. Thus, (c) implies that $\sum_{i<j} |t_{ij}|^2 = 0$, so T is diagonal. The factorization $A = UTU^*$ is equivalent to the identity $AU = UT$, which says that $Au_i = \lambda_i u_i$ for each $i = 1, \dots, n$. Thus, the n columns of U are orthonormal eigenvectors of A.* □

Problem 2 (25%): Sylvester equation.

Let us consider $A, B, C \in \mathcal{M}_n$.

1. Given a matrix $M \in \mathcal{M}_{m,n}$, define a vectorization procedure of M (Vec$(M) \in \mathbb{C}^{pn}$). Vectorize the equation $AX + XB = C$, $X \in \mathcal{M}_n$ and give the conidition for existence and uniqueness of the solution $\hat{X} \in \mathcal{M}_n$. Express Vec$(\hat{X})$ with the Kronecker product of A and B. **Correction:** *For a matrix $M = [c_1, \dots, c_n] \in \mathcal{M}_{m,n}$ with columns $c_j \in \mathbb{C}^m$, $j = 1, \dots, n$, we define*
$$Vec(M) := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{C}^{mn}.$$
The vectorial form of the equation is $G\,Vec(\hat{X}) = Vec(C)$ with $G \equiv I_n \otimes A + B \otimes I_n$. The existence and uniqueness of $\hat{X}$ is a consequence of the invertibbility of G which happens if and only if A and $-B$ do not have common eigenvalue. In that case: $Vec(\hat{X}) = G^{-1}\,Vec(C)$. □

2. Show that $Z : t \mapsto e^{tA}Ce^{tB}$ is solution to the differential equation:
$$\begin{cases} \dfrac{dZ}{dt} = AZ + ZB \\ Z(0) = C \end{cases}$$

Correction: *As the solution to such differential equations exists and is unique, let us simply check that:*

$$\frac{dZ}{dt} = Ae^{tA}Ce^{tB} + e^{tA}CBe^{tB} = Ae^{tA}Ce^{tB} + e^{tA}Ce^{tB}B = AZ + ZB, \tag{2}$$

since B and e^B commute. Of course one also has $Z(0) = e^{0A}Ce^{0B} = C$. □

3. Given $d \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ let us denote $J_d(\lambda)$ the Jordan block defined as:

$$J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \lambda \end{pmatrix} \in \mathcal{M}_d,$$

express $e^{tJ_d(\lambda)}$. **Correction:** $e^{tJ_d(\lambda)} = e^{t\lambda} \begin{pmatrix} 1 & t & & \frac{t^{d-1}}{(d-1)!} \\ & \ddots & \ddots & \\ & & \ddots & t \\ (0) & & & 1 \end{pmatrix}$ □

4. Deduce that when A and B only have strictly negative eigenvalues, $\hat{X} = -\int_0^\infty e^{tA}Ce^{tB}dt$ (we assume that this integral is well defined). **Correction:** *Integrating (2), we know that:*

$$Z(0) - \lim_{t\to\infty} Z(t) = A\left(-\int_0^\infty e^{tA}Ce^{tB}dt\right) + \left(-\int_0^\infty e^{tA}Ce^{tB}dt\right)B. \tag{3}$$

Recalling that $Z(0) = C$, we are simply left to show that $\lim_{t\to\infty} Z(t) = 0$. Let us then introduce $A = P^{-1}J_AP$ and $B = Q^{-1}J_BQ$, the Jordan decomposition of A and B, where $P, Q \in \mathcal{M}_n$ are invertible matrices and $J_A, J_B \in \mathcal{M}_n$ are block diagonal matrices with Jordan blocks on the diagonal. We see directly from Item 3 that if $\lambda < 0$, then:

$$\lim_{t\to\infty} e^{tJ_d(\lambda)} = \lim_{t\to\infty} e^{t\lambda} \begin{pmatrix} 1 & t & & \frac{t^{d-1}}{(d-1)!} \\ & \ddots & \ddots & \\ & & \ddots & t \\ (0) & & & 1 \end{pmatrix} = 0.$$

The same way, since J_A, J_B diagonal entries are strictly negative (they are the eigenvalues of A, B):

$$\lim_{t\to\infty} e^{tA} = \lim_{t\to\infty} P^{-1}e^{tJ_d(\lambda)}P = 0 \qquad \text{and} \qquad \lim_{t\to\infty} e^{tB} = 0.$$

Finally one has the identity:

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} e^{tA}Ce^{tB} = 0,$$

which allows us to conclude thanks to (3). □

5. Show that if A is Hermitian, then e^A is also Hermitian. Show that if A is positive semi-definite then e^A is also positive semi definite. **Correction:** *It is a clear consequence of the definition of the exponential:*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

□

6. Show that if $B = A^*$ and $-C$ is Hermitian positive semi-definite then X is also positive semidefinite.
Correction: *We already saw in the course that if $B = A^*$ and C Hermitian, then $\hat{X}$ is also Hermitian. If moreover C is positive semidefinite, then for all $u \in \mathbb{C}^n$:*
$$u^*\hat{X}u = -\int_0^\infty u^* e^{tA} C e^{tA^*} u dt \geq 0.$$
□

Problem 3 (10%): Schur complement.

1. Let us consider a matrix $A_{11} \in \mathcal{M}_n$, $A_{12} \in \mathcal{M}_{n,p}$, $A_{12} \in \mathcal{M}_{p,n}$ and $A_{22} \in \mathcal{M}_{p,p}$. Assuming that A_{11} is invertible, compute the product
$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix},$$
and deduce that $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ is invertible if and only if its so called "Schur complement" $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is invertible.
Correction: *A simple computation gives us:*
$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}, = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$
□

2. Given $X \in M_{p,q}$ let us introduce
$$K = \begin{bmatrix} I_p & X \\ X^* & I_q \end{bmatrix} \in M_{p+q}.$$
Show that K is positive definite if and only if X is a strict contraction (its singular values are all strictly lower than 1).
Correction: *We see from the Schur identity that K is positive definite iif $I_q - X^*X \succ 0$ which is again equivalent to $I_q \succ X^*X$ and $I_q \geq \sigma_1(X^*X)$, where σ_1 is the biggest singular value of X.* □

3. Given two positive semidefinite matrices $A, B \in \mathcal{M}_n$, show that the three following properties are equivalent:
   (a) $A \succ B$
   (b) $\rho(A^{-1}B) \leq 1$
   (c) There exists a contraction $X \in \mathcal{M}_n$ such that $B = A^{\frac{1}{2}}XA^{\frac{1}{2}}$.

   Correction: *Assume (a), then $I_n \succ A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $1 \geq \sigma_1(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$ which implies (C) with $X \equiv A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Besides note that since B, A are Hermitian:*
$$\sigma_1(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) = \lambda_{\max}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) = \lambda_{\max}(A^{-1}B),$$
which provides (b) – it is actually an equivalence. Now if we assume (c):
$$A^{-1}B = A^{-1}A^{\frac{1}{2}}XA^{\frac{1}{2}} = A^{-\frac{1}{2}}XA^{\frac{1}{2}},$$
which means that $A^{-1}B$ is similar to a contraction, therefore $\sigma_1(A^{-1}B) \leq 1$ and one can deduce (a). □

4. Let $H = \left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right) \in M_{p+q}$ be Hermitian with $A \in M_p$ and $C \in M_q$. Show the equivalence:
   (a) H is positive definite.
   (b) A is positive definite and $C - B^*A^{-1}B$ is positive definite.
   (c) A and C are positive definite and $\rho(B^*A^{-1}BC^{-1}) < 1$.

   Correction: *Simple consequence of the previous results.* □