# Homework 2: Jordan decomposition and exponential of matrices

## General instructions

- You must submit your results and solutions in a single PDF file. We recommend using LaTeX with a basic template, which you can reuse for each homework assignment.
- This assignment is due on March 8th 2024 by 23:00:00. Submissions will not be accepted after this deadline.
- If you have any questions, you can contact TA Zhang Wenrang via email at 223040237@link.cuhk.edu.cn, or during their office hour (Wednesday 14:00-15:00, ZhiXin 4F, seat 72).

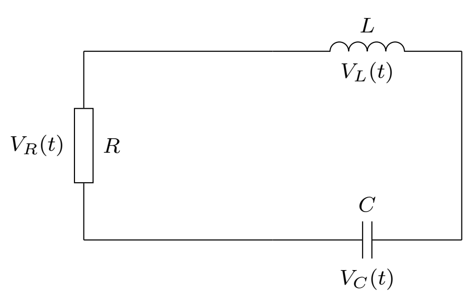
## Problem 1: Jordan decomposition (10*3 = 30 points)

The goal of this problem is to write the Jordan decomposition of the matrix:

$$B = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

1. Compute the characteristic polynomial of $B$ and deduce the possible Jordan decompositions of $B$
2. Express the eigenspaces of $B$ and provide the algebraic and geometric multiplicities of the eigenvalues of $B$.
3. Find two vectors $v_2, w_3 \in \ker(B - I_3)^2$ such that $Bv_2 = v_2$ and $Bw_3 = v_2 + w_3$. Deduce a change of basis matrix $P$ that allows to find the Jordan decomposition of $B$ (no need to invert $P$).

## Problem 2: Dynamic of an electrical network (10*2 = 20 points)




Noting $I$ the intensity in the circuit, one has the following relations that model the dynamic behavior of the circuit:

$$\begin{aligned} L\frac{dI}{dt} &= V_L, \\ C\frac{dV_C}{dt} &= -I, \\ RI &= V_R, \\ V_L - V_C + V_R &= 0. \end{aligned}$$

1. Express matricially the differential system of equation of $y = \begin{pmatrix} I \\ V_C \end{pmatrix}$. [Hint] Establish an equation between $\frac{dy}{dt}$ and $y$.
2. Solve the system with initial condition $y(0) = \begin{pmatrix} I_0 \\ 0 \end{pmatrix}$. We want a result without exponential of matrices (exponentials on the entries of a matrix are allowed).

## Exercises (10*5 = 50 points)

1. Given $A \in \mathcal{M}_n$, $B \in \mathcal{M}_m$ and $C \in \mathcal{M}_{n,m}$ such that $\text{Rank}(C) = m$ and $AC = CB$, prove that all the eigen values of $B$ are eigen values of $A$.
2. Express the sufficient and necessary conditions on the coefficients $a, b, c, d \in \mathbb{R}$ such that the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is triangularizable in $\mathcal{M}_n(\mathbb{R})$. Same question for $A$ to be diagonalizable in $\mathcal{M}_n(\mathbb{R})$.
3. Find all the canonical forms that can have a matrix $A$ having the characteristic polynomial:
$$\chi_A = (X-2)^3(X+1)^2$$
4. What is a necessary condition of two matrices $A$, $B$ such that $e^{A+B} \neq e^A e^B$ (no need for proof)? Give one example pair of such matrices for $A, B \in \mathcal{M}_2$.
5. Prove that for any $A \in \mathcal{M}_n(\mathbb{C})$, $\det(e^A) = e^{\text{Tr}(A)}$.