Homework 4: MUSIC algorithm

General instructions

- You must submit your results and solutions in a single PDF file. We recommend using LaTeX with a basic template, which you can reuse for each homework assignment.
- This assignment is due on 07/03/2024, by 11:59 PM. Submissions will not be accepted after this deadline.
- If you have any questions, you can contact TA Zhang Wenrang via email at 223040237@link.cuhk.edu.cn, or during office hours on Wednesday from 14:00 to 15:00 in room ZX, 4F-72.

Problem 1 (50%)

Let $A, B \in \mathcal{M}_n$. We want to prove:

$$A \text{ is unitarily similar}^1 \text{ to } B \iff \exists S \in M_n, \text{ invertible, such that } A = SBS^{-1} \text{ and } A^* = SB^*S^{-1}. \qquad (*)$$

1. Given a matrix $M \in \mathcal{M}_n$, non singular, show that there exist unique matrices $P, Q \in M_n$ positive semidefinite and $U \in M_n$ unitary such that $M = PU = UQ$. Show in passing that P is a polynomial in MM^* and Q is a polynomial in M^*M. *(**Hint:** recall from the proof of the theorem on the existence and uniqueness of roots of Hermitian matrices that given any H positive definite, $H^{\frac{1}{2}}$ is a polynomial of H).* **Correction:** *Considering the singular decomposition of M, we know that there exist unitary matrices $V \in \mathcal{M}_n$ and $W \in \mathcal{M}_n$, and a positive definite diagonal matrix $\Sigma \in \mathcal{M}_n$, such that $M = V\Sigma W^*$. Note then that $M = (V\Sigma V^*)(VW^*) = PU$, in which $P = V\Sigma V^*$ is positive semidefinite and $U = VW^*$ is unitary. Since $P^2 = V\Sigma\Sigma V^* = V\Sigma\Sigma^T V^* = (V\Sigma W^*)(W\Sigma^T V^*) = MM^*$, P is uniquely determined as the positive semidefinite square root of MM^*. As a consequence, $U = P^{-1}M$ is also uniquely determined. One deduce the same way that $M = UQ$ with $Q \equiv W\Sigma W^*$ that satisfies $Q^2 = M^*M$, the uniqueness is then deduced similarly. As square root of, respectively, MM^* and M^*M, P and Q are polynomials of, respectively, MM^* and M^*M.* □

2. Show the implication " $\implies$ " in $(*)$ **Correction:** *If one assumes that A is unitarily similar to B, that means that there exists a unitary matrix $U \in \mathcal{M}_n$ such that $A = U^*BU$, then one also has $A^* = U^*B^*U$, denoting $S \equiv U^*$, one retrieve the right-hand side of $(*)$.* □

3. Assuming from now on the right assertion in $(*)$, show that $A(SS^*) = (SS^*)A$. **Correction:** *Simply note that $ASS^* = SBS^* = (SB^*S^*)^* = (SB^*S^{-1}SS^*)^* = (A^*SS^*)^* = SS^*A$.* □

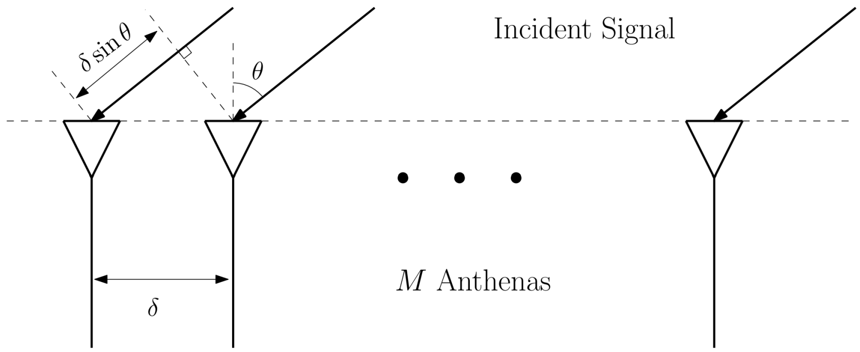
4. Let $S = PU$ be the decomposition introduced in Item 1, explain why $AP = PA$. **Correction:** *Recall from Item 1 that P is a polynomial of SS^* and from Item 3 that SS^* commutes with A, therefore P commutes with A.* □

5. Conclude on the validity of the converse implication "$\impliedby$". **Correction:** *Injecting $S = PU$ in the identity $A = SBS^{-1}$, one obtains $A = PUBU^*P^{-1}$ which implies $UBU^* = P^{-1}AP = A$ thanks to Item 4.* □

[1] It means that there exists a unitary matrix $U \in \mathcal{M}_n$ such that $U^*AU = B$

Problem 2 (50%) (Direction of Arrival (DoA) estimation problem by MUSIC)

Consider a wireless scenario where a multiple-antenna receiver receives signals from K sources with different directions of arrivals. One incident signal coming from an angle θ can be represented followingly:



We want to retrieve the angles of the different signals. Let M be the number of antennas (it will replace here the time window "d" that we introduced in the course), $(s_k[n])_{n\in[N]} \in \mathbb{C}^N$ denote the k^{th} source signal sequence, $k = 1, \ldots, K$, which are assumed to be independent from one another. The received signal vector can be expressed as

$$\forall n \in [N] : \quad y[n] = \sum_{k=1}^{K} a(\theta_k)s_k[n] + w[n] \in \mathbb{C}^M$$

where $w[n] \sim \mathcal{N}(0, \sigma_w I_M)$ is a noise vector $a(\theta_k)$ is the steering vector of source k with $\theta_i \in [-\pi/2, \pi/2]$ (rad) and

$$a(\theta_k) = \begin{bmatrix} 1 \\ e^{\mathbf{i}\pi\delta\sin(\theta_k)} \\ \vdots \\ e^{\mathbf{i}(M-1)\pi\delta\sin(\theta_k)} \end{bmatrix} \in \mathbb{C}^M.$$

1. Show that the correlation matrix $R_y[n] = \mathbb{E}[y[n]y[n]^*]$ can be expressed as

   $$R_y[n] = A\mathbb{E}[S[n]]S[n]^*]A^* + \sigma_w^2 \mathbf{I}_M,$$

   where $A = [a(\theta_1), a(\theta_2), \ldots, a(\theta_K)] \in \mathcal{M}_{M,K}$ and $S[n] = [s_1[n], \ldots, s_K[n]] \in \mathbb{C}^K$.

   We will now assume that

   $$\mathbb{E}[S[1]S[1]^*] \approx \cdots \approx \mathbb{E}[S[N]S[N]^*] \approx \frac{1}{N}SS^*$$

   where $S = [S[1], \ldots, S[N]] \in \mathcal{M}_{K,N}$ and that $\frac{1}{N}SS^*$ is of rank $K \leq N, M$. Following the idea of MUSIC presented in the lecture and noting that:

   $$R_y[1] \approx \cdots \approx R_y[N] \approx \hat{R}_y = \frac{1}{N}\sum_{n=1}^{N} y[n]y[n]^*,$$

   illustrate how the MUSIC can be applied to here for estimating the DoAs $\theta_1, \ldots, \theta_K$.
   Correction: *Denote $y[n] = AS[n] + w[n]$ where*

   $$A = \begin{bmatrix} a(\theta_1) & a(\theta_2) & \cdots & a(\theta_K) \end{bmatrix} \qquad S[n] = \begin{bmatrix} s_1[n] \\ s_2[n] \\ \vdots \\ s_K[n] \end{bmatrix}$$

We have

$$\begin{aligned} R_y[n] = \mathbb{E}[y[n]y[n]^*] &= \mathbb{E}[(AS[n] + w[n])(S[n]^*A^* + w[n]^*)] \\ &= \mathbb{E}[AS[n]S[n]^*A^*] + \mathbb{E}[AS[n]w[n]^*] + \mathbb{E}[w[n]S[n]^*A^*] + \mathbb{E}[w[n]w[n]^*] \end{aligned}$$

$w[n]$ *is white with 0 mean and correlation matrix* $\sigma_w^2 I_M$*, so*

$$R_y[n] = \mathbb{E}[AS[n]S[n]^*A^*] + \sigma_w^2 I_M$$

A *is invariant, so* $\mathbb{E}[AS[n]S[n]^*A^*] = A\mathbb{E}[S[n]S[n]^*]A^*$*. Therefore*

$$\boxed{R_y[n] = A\mathbb{E}[S[n]S[n]^*]A^* + \sigma_w^2 I_M}$$

We assume $\mathbb{E}[S[1]S[1]^*] \approx \cdots \approx \mathbb{E}[S[N]S[N]^*] \approx \frac{1}{N}SS^*$ *hence*

$$R_y[n] \approx \frac{1}{N}ASS^*A^* + \sigma_w^2 I_M \quad \Longrightarrow \quad \boxed{\hat{R}_y = \frac{1}{N}\sum_{n=1}^{N} y[n]y[n]^* \approx \frac{1}{N}ASS^*A^* + \sigma_w^2 I_M}$$

.. We want to find $\{\theta_k\}_k$ *starting from* $\hat{R}_y$ *which we can construct with the received signals. Analogically with MUSIC from the lecture, denote* $z_i = e^{\mathbf{i}\pi\delta\sin(\theta_i)}$*, we have*

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_K \\ \vdots & \vdots & \cdots & \vdots \\ z_1^{M-1} & z_2^{M-1} & \cdots & z_K^{M-1} \end{bmatrix}$$

A *is Vandermonde matrix. Since* $K \le M$*,* A *is always full column rank. Thus* A^* *is full row rank.* $\sigma_k^2 > 0$ *for all* k*, we know* SS^* *is invertible. So Range*$(ASS^*A^*) =$ *Range*(A)*.*
ASS^*A^* *is Hermitian. Denote the EVD of* ASA^* *as*

$$ASS^*A^* = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}$$

where $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ *is unitary and* Λ *is real-valued diagonal.*
We know $\{a(\theta_k)\}_k$ *span Range*(ASS^*A^*)*, therefore Span*$\{a(\theta_1), \cdots, a(\theta_K)\} =$ *Range*(V_1)*. Then*

$$\begin{aligned} \hat{R}_y &= \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} + \sigma_w^2 VV^* \\ &= \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 + \sigma_w^2 I_K & \\ & \sigma_w^2 I_{M-K} \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} \end{aligned}$$

$\boxed{\textit{\textbf{Solution}}}$*: we can do EVD of* R_y *and find* V_2*, since* V_2 *should correspond to some significantly small eigenvalues. Then with a dense enough discrete sequence of all* $\theta \ge 0$ *we calculate*

$$f(\theta) = \frac{1}{\|a(\theta)^*V_2\|}$$

$\boxed{\textit{if some } \theta \in \{\theta_k\}_k \textit{ that we want to find, we will see a peak on } f(\theta)}$. □

2. Set $K = 3$, $\theta_1 = 0°$, $\theta_2 = 30°$, $\theta_3 = -60°$, $M = 10$, $N = 500$, $\sigma_w^2 = 1$, that for all $k \in K$, $s_k[n] \sim \mathcal{N}(0, \sigma_s)$ with $\sigma_s = 1$ and that $s_k[1], \ldots, s_k[N]$ are all independent from one another. After generating the sequences $s_k[n]$ and $w[n]$, $n = 1, \ldots, N$, implement MUSIC algorithm and illustrate its performance via appropriate plots. Try different values of K, M, N and σ_w and discuss how these parameters affect the algorithm performance.

For the second item, please print your codes and append them to your HW solution report. Both the logic of the presented discussion and readability of your codes will be take into account for the grade.