Lecture 1

Matrix analysis: general notions

1 Notations

- $\mathbb{N}$: set of positive integers: $\mathbb{N} = \{0, 1, 2, \ldots\}$. Below we consider $n, p, m \in \mathbb{N}$.
- $[n]$: $[n] = \{1, \ldots, n\}$.
- $\mathfrak{S}_n$: the symmetric group: the set of permutations of elements of $[n]$ (i.e. the set of all the bijections from $[n]$ to $[n]$, there are $n!$ permutation in total). When we want the symmetric group to act on a given finite set I different from $[n]$, one can note $\mathfrak{S}_I$ the set of permutation of this set. Note that $\mathfrak{S}_n = \mathfrak{S}_{[n]}$.
- $\mathrm{Sgn}(\sigma)$: Signature of a permutation $\sigma \in \mathfrak{S}_n$; it is equal to $+1$ if the permutation σ can be obtained with an even number of transpositions (exchanges of two elements) otherwise, it is equal to -1.
- Given two sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$, we write $a_n = O(b_n)$ if there exists a constant $C > 0$ and an integer n_0 such that whenever $n \geq n_0$ we have:

$$a_n \leq C b_n.$$

- $\mathbb{R}$: Real valued space.
- $\mathbb{C}$: Complex-valued space.
- $\mathbb{K}$: Either $\mathbb{R}$ or $\mathbb{C}$.
- $\delta_{x,y}$: for $x, y \in \mathbb{C}$, $\delta_{x,y} = \begin{cases} 0, & x \neq y, \\ 1, & x = y, \end{cases}$.
- $\mathbb{R}^n$ (resp. $\mathbb{C}^n$): n-dimensional real (resp. complex) space.
- $\mathcal{M}_{p,n}$ (resp. $\mathcal{M}_{p,n}(\mathbb{R})$): Set of $p \times n$ complex (resp. real) matrices, $\mathcal{M}_p = \mathcal{M}_{p,p}$. Some authors use the notations $\mathbb{C}^{p\times n}$ and $\mathbb{R}^{p\times n}$ that it are useful to know. We note $\mathbf{0}$ the null matrix of $\mathcal{M}_{p,n}$.
- Any element $x \in \mathbb{R}^n$ (or $x \in \mathbb{C}^n$) is identified with a column matrix of $\mathcal{M}_{1,n}$ (one calls that column vector). **There are no "row vectors".** We note $x_1, \ldots, x_n$ its entries:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

- Let $I_n \in \mathcal{M}_n$ be the identity matrix:

$$I_n = \begin{pmatrix} 1 & & (0) \\ & \ddots & \\ (0) & & 1 \end{pmatrix}.$$

- Given $i \in [n]$, let us note e_i the i-th column of I_n, i.e., $I_n = [e_1, \dots, e_n]$. The dimension n of each $(e_i)_{i\in[n]}$ is not specified in the notation, it can be deduced from the context.

- Given a matrix $A \in \mathcal{M}_{p,n}$, we note $(A_{i,j})_{i\in[p],j\in[n]} \in \mathbb{C}^p$ its entries, $A_{\cdot 1}, \dots, A_{\cdot n} \in \mathcal{M}_{p,1}(\mathbb{C})$ (or $a_1, \dots, a_n \in \mathbb{C}^p$) its columns and $A_{1\cdot}, \dots, A_{p\cdot} \in \mathcal{M}_{1,n}(\mathbb{C})$ its rows:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{p,1} & \cdots & A_{p,n} \end{pmatrix} = \begin{pmatrix} A_{\cdot 1}, \dots, A_{\cdot n} \end{pmatrix} = \begin{pmatrix} a_1 \cdots a_n \end{pmatrix} = \begin{pmatrix} A_{1\cdot} \\ \vdots \\ A_{p\cdot} \end{pmatrix}.$$

- The transpose and the hermitian transpose of $A \in \mathcal{M}_{p,n}$ are respectively noted:

$$A^T = \begin{pmatrix} A_{1,1} & \cdots & A_{p,1} \\ \vdots & & \vdots \\ A_{1,n} & \cdots & A_{p,n} \end{pmatrix} \in \mathcal{M}_{n,p} \qquad \text{and} \qquad A^* = \begin{pmatrix} \bar{A}_{1,1} & \cdots & \bar{A}_{p,1} \\ \vdots & & \vdots \\ \bar{A}_{1,n} & \cdots & \bar{A}_{p,n} \end{pmatrix} \in \mathcal{M}_{n,p},$$

  where for any $z \in \mathbb{C}$, $\bar{z} \equiv \Re(z) - \Im(z)\mathbf{i}$ is the complex conjugate of z.

- **Matrix Product:** Given matrices $A \in \mathcal{M}_{p,m}$ and $B \in \mathcal{M}_{m,n}$, the matrix product AB is defined as a matrix $C \in \mathcal{M}_{p,n}$ with entries:

$$C_{i,j} = \sum_{k=1}^{m} A_{i,k} B_{k,j}.$$

  Note the simple expressions, for any $i \in [p]$ and $j \in [n]$:

$$C = \sum_{k=1}^{m} A_{\cdot k} B_{k\cdot}, \qquad C_{\cdot j} = AB_{\cdot j}, \qquad \text{and} \qquad C_{i\cdot} = A_{i\cdot} B,$$

- Given $i \in [p], j \in [n]$: $E_{i,j} \equiv e_i e_j^T = [\ \underbrace{0, \dots, 0}_{j-1 \text{ columns}}\ , e_i,\ \underbrace{0, \dots, 0}_{n-j \text{ columns}}\ ] \in \mathcal{M}_{p,n}$. The dimension $p \times n$ of each $(E_{i,j})_{i\in[p],j\in[n]}$ is not specified in the notation, it can be deduced from the context.

- The trace of a square matrix A, denoted by $\mathrm{Tr}(A)$, is the sum of its diagonal elements:

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{i,i}.$$

2 Definitions

- **Vector Space:** Given a scalar field $\mathbb{K}$, a set V endowed with a sum and a scalar product with elements of $\mathbb{K}$ is said to be a vector space iif the following properties are satisfied:

  1. **Stable through Addition:** For any two vectors x and y in V, the sum $x + y$ is also in V.
  2. **Stable through Scalar Multiplication:** For any scalar $\alpha \in \mathbb{K}$ and vector x in V, the product αx is also in V.
  3. **Zero Vector:** There exists a zero vector $\mathbf{0}$ in V such that $x + \mathbf{0} = x$ for any vector x in V.

4. **Additive Inverse:** For every vector x in V, there exists an additive inverse $-x$ such that $x + (-x) = \mathbf{0}$.

- **Linear Combination:** Let $x_1, x_2, \ldots, x_n$ be n vectors of a vector space V. A linear combination of these vectors is an expression of the form:

$$y = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n,$$

where $c_1, c_2, \ldots, c_n$ are scalars. By definition of a vector space, $y \in V$.

- **Subspace of a Space** Let V be a vector space. A subset U of V is called a subspace of V if it is itself a vector space with respect to the vector space operations of V.

- **Sum of subspaces:** Given two subspaces F, G, the sum $F + G$ is the subspace:

$$F + G = \{x + y, x \in F, y \in G\}.$$

If for any $x \in F, y \in G$ $x + y = 0 \Rightarrow x = y = 0$, the sum is said to be "direct", and one usually denote $F \oplus G$ instead of $F + G$.

- **Span of Vectors:** The span of a set of vectors $\{v_1, v_2, \ldots, v_n\}$, denoted by $\text{Span}(v_1, v_2, \ldots, v_n)$, is the set of all possible linear combinations of these vectors. If one work with the scalar field $\mathbb{K}$, one will also use the notation:

$$\text{Span}(v_1, v_2, \ldots, v_n) = \mathbb{K}v_1 + \cdots + \mathbb{K}v_n$$

- **Generative Families of Vectors:** A family of vectors $\{v_1, v_2, \ldots, v_n\}$ is said to be a generative family if the span of this family is the entire vector space.

- **Linearly Independent:** A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is said to be linearly independent if the only solution to the equation $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \mathbf{0}$ is the trivial solution $c_1 = c_2 = \ldots = c_n = 0$.

- **Orthogonal family:** A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is said to be *orthogonal* if each pair of distinct vectors is orthogonal, i.e., $v_i^* v_j = 0$ for all $i \neq j$.

- **Basis:** A basis of a vector space V is a linearly independent generative family of vectors. In other words, a set of vectors $\{v_1, v_2, \ldots, v_n\}$ is a basis for V if it spans V and is linearly independent.

- **Dimension of a Subspace:** The dimension of a subspace U, denoted by $\dim(U)$, is the maximum number of linearly independent vectors in U. It is also the number of vectors in any basis for U (see Subsection 3.1, Item 3).

- **Norm:** Let V be a vector space over the field of real or complex numbers. A norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying the following properties for all vectors $u, v \in V$ and all scalars $\alpha \in \mathbb{R}$ or $\mathbb{C}$:

  1. **Non-negativity:** $\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = \mathbf{0}$ (the zero vector).
  2. **Scalar Multiplication:** $\|\alpha u\| = |\alpha| \|u\|$.
  3. **Triangle Inequality:** $\|u + v\| \leq \|u\| + \|v\|$.

  A vector space equipped with a norm is called a normed vector space. **Euclidean Norm:**

  The Euclidean norm (or 2-norm) of a vector $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{C}^n$, denoted as $\|v\|$ or $\|v\|_2$, is defined as:

$$\|v\| = \sqrt{|v_1|^2 + |v_2|^2 + \ldots + |v_n|^2} = \sqrt{v^* v}.$$

Given $p > 0$, the $\|\cdot\|_p$ norm is defined as:

$$\|v\|_p = \left(|v_1|^p + |v_2|^p + \ldots + |v_n|^p\right)^{\frac{1}{p}}.$$

The $\|\cdot\|_\infty$ norm is defined as:

$$\|v\|_\infty = \max\left(v_1, v_2, \ldots, v_n\right).$$

- **Orthogonal Complement:** For a subspace $S \subseteq \mathbb{R}^m$ the *Orthogonal Complement* is the subspace of $\mathbb{R}^n$ defined as:
$$S^\perp = \{y \in \mathbb{R}^m : \Re(y^* x) = 0, \forall x \in S\}.$$

- **Orthonormal Basis:** A basis $\{v_1, v_2, \ldots, v_n\}$ of a vector space V is called an *orthonormal basis* if it is orthogonal and each of its vectors is of unit length, i.e., $\|v_i\| = 1$ for all i.

- **Coordinates of a vector in a basis and representation of a matrix in a basis:**

  Let v be a vector in a vector space V, and let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. The coordinates of v in the basis $\mathcal{B}$, denoted as $[v]_\mathcal{B}$, are the unique scalars $c_1, c_2, \ldots, c_n$ such that:

  $$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.$$

  The vector v can be represented as a column vector in terms of its coordinates in the basis $\mathcal{B}$:

  $$[v]_\mathcal{B} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

  Given a matrix $A = \in \mathcal{M}_{p,n}$, the representation of A in $\mathcal{B}$ is defined as:

  $$[A]_\mathcal{B} \equiv \left([Av_1]_\mathcal{B}, \ldots, [Av_n]_\mathcal{B}\right).$$

- **Elementary matrices:** We consider below $i, j \in [n]$ such that $i \neq j$ and $\lambda \in \mathbb{K}$:

  1. *Row/column Swap Matrix*
  $$P_{ij} := [e_1, \ldots, e_{i-1}, e_j, e_{i+1}, \ldots, e_{j-1}, e_i, e_{j+1}, \ldots, e_n] \in \mathcal{M}_n.$$
  2. *Row/column Scaling Matrix*
  $$M_i(\lambda) := [e_1, \ldots, e_{i-1}, \lambda e_i, e_{i+1}, \ldots, e_n] \in \mathcal{M}_n.$$
  3. *Row/column Addition Matrix*
  $$G_{ij}(\lambda) := I_n + \lambda E_{ij} = [e_1, \ldots, e_{i-1}, e_i + \lambda e_j, e_{i+1}, \ldots, e_n] \in \mathcal{M}_n.$$

- **Image of a Matrix (or Range Space):** For a matrix A, the image, or range space, denoted by $\mathrm{Im}(A)$, is the set of all possible linear combinations of the columns of A.

- **Rank of a Matrix:** The rank of a matrix A, denoted by $\mathrm{Rk}(A)$, is the maximum number of linearly independent columns (or rows) in A. It is equal to the dimension of the column space (or row space) of A.

- **Kernel (or Null Space):** The kernel, or null space, of a matrix A, denoted by $\ker(A)$, is the set of all vectors x such that $Ax = \mathbf{0}$.

- **Invertible Matrix:** A matrix $A \in \mathcal{M}_n$ is said to be invertible or non-singular if there exists an inverse matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$.

- **Singular Matrix:** A square matrix A is said to be singular if it is not invertible. In other words, A is singular if there exists no inverse matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.
- **Injective/Surjective Matrix:** A rectangular matrix $A \in \mathcal{M}_{p,n}(\mathbb{K})$ is said to be injective if $\mathrm{Ker}(A) = \{0\}$. It is said to be surjective if $\mathrm{Im}(A) = \mathbb{K}^n$.
- **Eigenvalue and Eigenvector:** Given $A \in \mathcal{M}_n$, a scalar λ is called an eigenvalue of A if there exists a non-zero vector v, called an eigenvector, such that:
$$Av = \lambda v.$$
The eigenspace E_λ associated to λ is the set of eigenvectors associated to lambda, it is exactly:
$$E_\lambda = \mathrm{Ker}(A - \lambda I_n).$$
- **Spectrum, spectral radius:** The set of all $\lambda \in \mathbb{C}$ that are eigenvalues of $A \in \mathcal{M}_n$ is called the *spectrum* of A and is denoted by $\mathrm{Sp}(A)$. If not specified, $\mathrm{Sp}(A)$ is the set of complex eigenvalues of A (that is always non-empty unlike the set of real eigenvalues, see Subsection 3.8, Item 3). The *spectral radius* of A is the nonnegative real number $\rho(A) = \max\{|X| : \mathrm{Sp}(A)\}$. This is just the radius of the smallest disc centered at the origin in the complex plane that includes all the eigenvalues of A.
- **Triangular Matrix:** A square matrix T is said to be upper triangular if all its entries below the main diagonal are zero, i.e., $T_{ij} = 0$ for $i > j$. Similarly, T is said to be lower triangular if all its entries above the main diagonal are zero, i.e., $T_{ij} = 0$ for $i < j$.
- **Change of Basis Matrices:**
Let $\mathcal{B} = \{w_1, w_2, \ldots, w_n\}$ be a base for the vector space V^n. The change of basis[1] to $\mathcal{B}$ is performed thanks to a "change of basis matrix" P defined by:
$$P = (w_1, w_2, \ldots, w_n)\,,$$
It is invertible thanks to Subsection 3.5. Actually any invertible matrix can be associated to a basis and therefore be seen as a change of basis matrix.
- **Similarity Between Matrices:** Two square matrices A and B are said to be similar if there exists an invertible matrix P such that:
$$B = P^{-1}AP.$$
We will see in Subsection 3.5, Item 12 that B is actually the representation of A in the basis composed of the columns of P.
- **Diagonalizable Matrix:** A square matrix A is said to be diagonalizable if it is similar to a diagonal matrix (i.e., if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix or equivalently if there exists a basis $\mathcal{B}$ such that $[A]_\mathcal{B}$ is diagonalizable).
- **Triangularizable Matrices:** A square matrix A is said to be *triangularizable* if it is similar to an upper triangular matrix (i.e. there exists an invertible matrix P such that $P^{-1}AP$ is an upper triangular matrix or equivalently there exists a basis $\mathcal{B}$ such that $[A]_\mathcal{B}$ is triangular).
- **Orthogonal, unitary Matrices:** A square matrix $Q \in \mathcal{M}_n$ is called *orthogonal* if its transpose is equal to its inverse, i.e., $Q^TQ = QQ^T = I_n$, where I_n is the identity matrix. A square matrix $U \in \mathcal{M}_n(\mathbb{C})$ is called *unitary* if its conjugate transpose U^* is equal to its inverse.
- **Symmetric, Hermitian Matrices:** A matrix $P \in \mathcal{M}_n$ is called *symmetric* it it is equal to its transpose ($P^T = P$). For complex matrices, a square matrix $H \in \mathcal{M}_n(\mathbb{C})$ is called *Hermitian* if its conjugate transpose (adjoint) is equal to itself, i.e., $H^* = H$. Be careful, the two notions are not equivalent for complex matrices ($\mathbf{i}I_n$ is symmetric bit not hermitian).

[1] We will later see in Subsection 3.5, Item 12 that given a matrix $A \in \mathcal{M}_n$, $[A]_\mathcal{B} = P^{-1}AP$.

- **Positive Semidefinite Matrices:** A symmetric matrix P is said to be *positive semidefinite* if for any vector $x \neq \mathbf{0}$, the quadratic form $x^T P x \geq 0$. In the case of Hermitian matrices H, the condition is $x^* H x \geq 0$.
- **Nilpotent Matrices:** A matrix $A \in \mathcal{M}_n$ is said to be *nilpotent* if there exists an integer $r \geq 0$ such that $A^r = \mathbf{0}$.
- **Determinant of a Matrix:** Given $A \in \mathcal{M}_n$, we note $\det(A)$ or $|A|$, the determinant if A defined below with the Signature formula of Leibniz:

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{Sgn}(\sigma) A_{1,\sigma(1)} \cdots A_{n,\sigma(n)}$$

- **Minors:** The (i,j) minor of A will be noted $|A_{-i,j}|$,and is defined as the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A:

$$A_{-i,j} \equiv \begin{pmatrix} A_{1,1} & \cdots & A_{1,j-1} & A_{1,j+1} & \cdots & A_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{i-1,1} & \cdots & A_{i-1,j-1} & A_{i-1,j+1} & \cdots & A_{i-1,n} \\ A_{i+1,1} & \cdots & A_{i+1,j-1} & A_{i+1,j+1} & \cdots & A_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{p,1} & \cdots & A_{p,j-1} & A_{p,j+1} & \cdots & A_{p,n} \end{pmatrix}$$

- **Cofactor matrix or Comatrix:** Given a matrix $A \in \mathcal{M}_n$ let us introduce for all $i,j \in [n]$ the scalar $C_{i,j} = (-1)^{i+j}|A_{-i,j}|$, where $|A_{-i,j}|$ is the (i,j) minor of A, then the matrix $\operatorname{Com}(A) \equiv (C_{i,j})_{i,j \in [n]}$ is called the *comatrix* of A.
- **Monomial polynomial:** A *monomial* P is a polynomial of $\mathbb{K}[X]$ that lets only appear one exponent of X, P writes aX^k for some $a \in \mathbb{K}$, $k \in \mathbb{N}$.
- **Degree of a polinomial:** The degree of a polynomial P denoted "$\deg P$" is the highest exponent of the monomials appearing in the expression of P.
- **Monic polinomial:** A *monic polynomial* is a non-zero univariate polynomial (that is, a polynomial in a single variable) in which the leading coefficient (the nonzero coefficient of highest degree) is 1.
- **Elementary polynomial:** An elementary polynomial is a polynomial of degree 1 that writes $a_1 X + a_0$, for two scalars a_0, a_1, $a_1 \neq 0$.
- **Polynomial of matrices:** A polynomial $P = a_n X^n + \cdots + a_1 X + a_0 \in \mathbb{C}[X]$ applied on a matrix $A \in \mathcal{M}_n$ is the matrix:

$$P(A) = a_n A^n + \cdots + a_1 A + a_0 I_n \in \mathcal{M}_n.$$

- **Characteristic Polynomial of a Matrix:** The characteristic polynomial of $A \in \mathcal{M}_n$ is the polynomial χ_A given by[2]:

$$\chi_A(X) \equiv \det(XI_n - A) \in \mathbb{C}[X],$$

  if $A \in \mathcal{M}_n(\mathbb{R})$, $\chi_A(X) \in \mathbb{R}[X]$.
- **Minimal annihilating polynomial:** Noting $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, the nonzero monic polynomial in $\mathbb{K}[X]$ which annihilate A and has least degree is called the *minimal annihilating polynomial* of A in $\mathbb{K}[X]$.
- **Eigenvalue multiplicity:** Given a matrix A having an eigenvalue λ, the dimension of the eigenspace E_λ associated to λ is classically called the "geometric multiplicity" of λ and must be compared with the "algebraic multiplicity" of λ which is the number of times the factor $X - \lambda$ appears in the characteristic polynomial χ_A of A.

[2]Some authors rather define the characteristic polynomial as $\chi_A(X) \equiv \det(A - XI_n)$, but we prefer our choice that produces monic polynomial.

3 General properties.

3.1 Basis, dimension

In what follows, V designates a $\mathbb{K}$-vector space.

1. Given $k+1$ vectors $w, v_1, \dots, v_k \in V$, if $w = \sum_{i=1}^{k} \lambda_i v_i$ and $\lambda_i \neq 0$, then $\mathrm{Span}(w, v_2, \dots v_k) = \mathrm{Span}(v_1, v_2, \dots v_k)$. **Proof:** *One already knows that Span$(w, v_2, \dots v_k) \subset$ Span$(v_1, v_2, \dots v_k)$ since w is a linear combination of $v_1, \dots, v_k$, besides $v_1 = \frac{1}{\lambda_1}(w - \sum_{i=2}^{k} \lambda_i v_i)$ is a linear combination of $w, v_2, \dots v_k$ which concludes the proof.* □

2. Let $W = \{w_1, \dots, w_n\}$ and $U = \{u_1, \dots, u_m\}$ be finite subsets of a vector space, and let $w_1, \dots, w_n$ be linearly independent. If $W \subseteq \mathrm{Span}\{u_1, \dots, u_m\}$, then $n \leq m$, and n elements of U, if numbered appropriately, can be exchanged with n elements of W such that

$$\mathrm{Span}\{w_1, \dots, w_n, u_{n+1}, \dots, u_m\} = \mathrm{Span}\{u_1, \dots, u_n, u_{n+1}, \dots, u_m\}.$$

In other words free families of vectors spaces have always less elements than generative families.

Proof: *By assumption, w_1 can be expressed as a linear combination of $\{u_1, \dots, u_m\}$ with non-zero coefficients. Without loss of generality, after renumbering if necessary, we assume the coefficient of u_1 in this linear combination is non-zero. By Item 1, this gives us*

$$Span\{w_1, u_2, \dots, u_m\} = Span\{u_1, u_2, \dots, u_m\}.$$

Continuing this process, we assume that we have exchanged vectors $u_1, \dots, u_r$ with $w_1, \dots, w_r$ such that

$$Span\{w_1, \dots, w_r, u_{r+1}, \dots, u_m\} = Span\{u_1, \dots, u_m\}.$$

By assumption we have $w_{r+1} \in span\{u_1, \dots, u_m\}$, and thus

$$w_{r+1} = \sum_{i=1}^{r} \lambda_i w_i + \sum_{i=r+1}^{m} \lambda_i u_i$$

for some scalars $\lambda_1, \dots, \lambda_m$. One of the scalars $\lambda_{r+1}, \dots, \lambda_m$ must be nonzero (otherwise w_{r+1} would be in span$\{w_1, \dots, w_r\}$, which contradicts the linear independence of $w_1, \dots, w_m$). After an appropriate renumbering, we have $\lambda_{r+1} \neq 0$, and Item 1 yields:

$$Span\{w_1, \dots, w_{r+1}, u_{r+2}, \dots, u_m\} = Span\{u_1, \dots, u_m\},$$

Extending this to $r = n-1$, we have

$$Span\{w_1, \dots, w_n, u_{n+1}, \dots, u_m\} = Span\{u_1, \dots, u_m\},$$

which implies that $n \leq m$. □

3. All the basis of V have the same number of elements. Together with Item 2, one deduces that in a space of dimension n, free families have n or less than n elements and generative families have n or more than n elements. In particular, the dimension of $\mathbb{K}^n$ is well defined and equal to n ($\{e_1, \dots, e_n\}$ is a basis of $\mathbb{K}^n$). **Proof:** *Let us assume that we are given two basis $\{v_1, \dots, v_k\}, \{w_1, \dots, w_n\} \subset V$. Since both families are free and generative, Item 2 allows us to set that $n \leq k$ and $k \leq n$ which implies $n = k$. It is straight forward to show that $e_1, \dots, e_n$ is linearly independent and generative in $\mathbb{K}^n$, therefore $\dim(\mathbb{K}^n) = n$.* □

4. Given n linearly independent vectors $\{v_1, \dots, v_n\}$ in a vector space V, if $V \setminus \mathrm{Span}(v_1, \dots, v_n) \neq \emptyset$ then $\forall w \in V \setminus \mathrm{Span}(v_1, \dots, v_n)$, $\{v_1, \dots, v_n, w\}$ is free. **Proof:** *Considering $w \in V \setminus Span\{v_1, \dots, v_n\}$, one could show that $v_1, \dots, v_n, w$ are also linearly independent. Indeed if there exist $\alpha_1, \dots, \alpha_{n+1}$, such that $\alpha_1 v_1 + \dots + \alpha_{n+1} w = 0$, then:*

(a) *if* $\alpha_{n+1} \neq 0$, $w = \frac{-1}{\alpha_{n+1}}(\alpha_1 v_1 + \cdots + \alpha_n v_n) \in Span(v_1, \ldots, v_n)$ *which contradicts our hypothesis.*

(b) *if* $\alpha_{n+1} = 0$ *then* $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ *and therefore* $\alpha_1 = \cdots = \alpha_n = 0$ *by freeness of* $v_1, \ldots, v_n$. *That proves that* $\{v_1, \ldots, v_n, w\}$ *is free.*

□

5. We assume here that $\dim(V) = n$. A family of n vectors $\mathcal{B} = (v_1, \ldots, v_n)$ is free iif it is a basis. A family of n vectors $\mathcal{B} = (v_1, \ldots, v_n)$ is generative iif it is a basis.

   Proof:

   (a) *Let us assume that* $\mathcal{B}$ *is free, if* V *is different from* $Span(\mathcal{B}) \subset V$, *then considering* $v_{n+1} \in V \setminus Span(\mathcal{B})$, *we know from Item 4 that* $(v_1, \ldots, v_n, v_{n+1})$ *is free, which contradicts Item 3.*

   (b) *Let us assume that* $\mathcal{B}$ *is generative. If we assume that there exist some coefficient* $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ *such that* $\alpha_1 v_1 + \cdots \alpha_n v_n = 0$ *and a certain* $i \in [n]$ *such that* $\alpha_i \neq 0$. *Then* $v_i = \frac{1}{\alpha_i}(\sum_{\substack{j=0 \\ j \neq i}} \alpha_j v_j) \in Span(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$, *that would implies that* $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ *is generative which contradicts again Item 3*

   □

6. **Basis extension Theorem.** Any set $\{v_1, \ldots, v_k\} \subset V$ of $k \leq n$ linearly independent vectors can be extended to a basis of V. **Proof:** *Iteratively for* $l \in \{k+1, \ldots, n\}$, *one can consider* $v_l \in \mathbb{K}^n \setminus Span(v_1, \ldots, v_{l-1})$, *then if* $v_1, \ldots, v_{l-1}$ *are linearly independent, we know from Item 4 that* $v_1, \ldots, v_l$ *are also linearly independent. We continue the process until* $l = n$. □

7. Given two subspace V, U, if $V \subset U$ and $\dim(U) = \dim(V)$ then $U = V$. **Proof:** *Any basis* $\mathcal{B}$ *of* V *is a free family of* U *of* $\dim(U)$ *elements, therefore Item 5 allows us to set that it is a basis of* U, *which means* $U \subset Span(\mathcal{B}) = V$. □

8. Given two subspace in V, U, let us denote $\mathcal{B}_U = (u_1, \ldots, u_{\dim(U)}$ and $\mathcal{B}_V = (v_1, \ldots, v_{\dim(V)})$, respectively, a basis of U and V. If U and V are in direct sum then $\mathcal{B}' = (u_1, \ldots, u_{\dim(U)}, v_1, \ldots, v_{\dim(V)})$ is a basis of $U \oplus V$ (and $\dim(U) + \dim(V) = \dim(U+V)$). Conversely, if $\dim(U) + \dim(V) = \dim(U+V)$, then U and V are in direct sum. This property generalises to k subspaces $U_1, \ldots, U_k$.

   Proof:

   (a) *Let us assume that* U *and* V *are in directt sum. We already know that* $\mathcal{B}'$ *is generative* $U+V$, *let us then show that it is free. Assuming that there exists some coeffincients* $\alpha_1, \ldots, \alpha_{\dim(U)}, \beta_1, \ldots, \beta_{\dim(V)} \in \mathbb{R}$ *such that:*

   $$\alpha_1 u_1 + \cdots + \alpha_{\dim(U)} u_{\dim(U)} + \beta_1 v_1 + \cdots + \beta_{\dim(V)} v_{\dim(V)} = 0$$

   The fact that U *and* V *are in direct sum then allows us to deduce that:*

   $$\begin{aligned} \alpha_1 u_1 + \cdots + \alpha_{\dim(U)} u_{\dim(U)} &= 0 \\ \beta_1 v_1 + \cdots + \beta_{\dim(V)} v_{\dim(V)} &= 0, \end{aligned}$$

   which implies that:

   $$\alpha_1 = \cdots = \alpha_{\dim(U)} = 0 \qquad \text{and} \qquad \beta_1 = \cdots = \beta_{\dim(V)} = 0,$$

   which finally allows us to conclude that B' *is free, and therefore a basis of* $U+V$.

   (b) *Let us assume that* $\dim(U+V) = \dim(U) + \dim(V)$. *Then* $\mathcal{B}'$ *is generative of* $U+V$ *and has* $\dim(U) + \dim(V) = \dim(U+V)$ *elements, that allows us to conclude that it is a basis of* $U+V$ *thanks to Item 5. Now, given* $x \in U$ *and* $v \in V$ *such that* $x + y = 0$, *expressing* x *and* y *as a linear combination of elements of, respectively* $\mathcal{B}_U$ *and* $\mathcal{B}_V$, *allows us to express the sum* $x + y = 0$ *as a linear combination of elements of* $\mathcal{B}'$ *finally allowing us to show that* $x = y = 0$ *by freeness of* $\mathcal{B}'$

   □

9. 🕮 Given two subspaces U, V:

$$\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

10. 🕮 Two subspaces U, V are in direct sum if and only if $U \cap V = \{0\}$. **Proof:** *Consequence of Items 8 and 9.* □

11. 🕮☛ **Gram-Schmidt.** Given a basis $\{x_1, \ldots, x_n\}$ of $\mathbb{R}^n$, there exists an orthonormal basis $\{z_1, \ldots, z_n\}$ such that for all $k \in [n]$, $z_k \in \text{Span}(x_1, \ldots, x_k)$. In particular if one introduces the matrix $X = (x_1, \ldots, x_n)$ and $Z = (z_1, \ldots, z_n)$, there exists an upper triangular matrix T such that:

$$Z = XT.$$

Proof: *Define $y_1 \equiv x_1$ and choose*

$$z_1 = \frac{y_1}{\sqrt{\langle y_1, y_1 \rangle}}$$

to normalize z_1. Define $y_2 = x_2 - \langle x_2, z_1 \rangle z_1$ to ensure y_2 is orthogonal to z_1, and choose

$$z_2 = \frac{y_2}{\sqrt{\langle y_2, y_2 \rangle}}$$

so that z_2 is normalized and orthogonal to z_1. Continue similarly. After determining $z_1, \ldots, z_{k-1}$, let

$$y_k = x_k - \langle x_k, z_{k-1} \rangle z_{k-1} - \ldots - \langle x_k, z_1 \rangle z_1,$$

so that y_k is orthogonal to $z_1, \ldots, z_{k-1}$, and again normalize y_k to obtain z_k.

$$z_k = \frac{y_k}{\sqrt{\langle y_k, y_k \rangle}}$$

Continue until all the desired orthonormal vectors $z_1, \ldots, z_n$ have been produced. □

12. 🕮☛ **Dimension Theorem (Rank-Nullity Theorem):** Given $A \in \mathcal{M}_n(\mathbb{K})$:

$$\dim(\text{Im}(A)) + \dim(\text{Ker}(A)) = n.$$

Proof: *With Subsection 3.1, Item 6 (basis extension Theorem): take $(v_1, \ldots, v_p)$ a base of $\ker A \subseteq \mathbb{K}^n$. Now completing it in a base $(v_1, \ldots, v_p, v_{p+1}, \ldots, v_n)$ of $\mathbb{K}^n$, one can express:*

$$\begin{aligned} A(\mathbb{K}^n) &= A(\mathbb{K}v_1) + A(\mathbb{K}v_2) + \cdots + A(\mathbb{K}v_p) + A(\mathbb{K}v_{p+1}) + \cdots + A(\mathbb{K}v_n) \\ &= \mathbb{K}Av_{p+1} + \cdots + \mathbb{K}Av_n. \end{aligned}$$

Let us then show that $Av_{p+1}, \ldots, Av_n$ is free. It we assume that there exist $\alpha_{p+1}, \ldots, \alpha_n$ such that $\alpha_{p+1}Av_{p+1} + \cdots + \alpha_n Av_n = 0$, then it means that $\alpha_{p+1}v_{p+1} + \cdots + \alpha_n v_n \in \ker(A) \cap Span(v_{p+1}, \ldots, v_n) = \{0\}$, which then implies $\alpha_{p+1} = \cdots = \alpha_n = 0$. Finally, Item 7 allows us to set:

$$\dim(KerA) + \dim(ImA) = p + (n - (p+1) + 1) = n = \dim(\mathbb{K}^n).$$

□

3.2 Orthogonal complement

Let $S \subseteq \mathbb{K}^m$ be a subspace of $\mathbb{K}^n$:

1. 🕮 $S \cap S^\perp = \{0\}$. **Proof:** *Given $X \in S \cap S^\perp$, $\|x\|^2 = x^*x = 0$ thus $x = 0$.* □

2. 🕮☛ $\mathbb{K}^n = S \oplus S^\perp$ (direct sum) and therefore $\dim(S) + \dim(S^\perp) = n$, thanks to Subsection 3.1 Item 8. **Proof:** *It is a consequence of Gramm-Schmidt (Subsection 3.1, Item 11), we consider an orthonormal basis of S and we extend it to get an orthonormal basis of $\mathbb{K}^n$, then it is straightforward to see that all the added vectors belong to $S^\perp$ and $S \oplus S^\perp = \mathbb{K}^n$.* □

3. $(S^{\perp})^{\perp} = S$. **Proof:** *Given $x \in S$, for all $y \in S^{\perp}$, $x^* y = 0$, therefore $x \in (S^{\perp})^{\perp}$ and $S \subset (S^{\perp})^{\perp}$. Now since we know from Item 2 that $S \oplus S^{\perp} = S^{\perp} \oplus (S^{\perp})^{\perp}$, Subsection 3.1, Item 8 allows us to set that $\dim(S) = \dim((S^{\perp})^{\perp})$ and the inclusion $S \subset (S^{\perp})^{\perp}$ allows us to conclude with Subsection 3.1, Item 7.* $\square$

4. If $S_1 \subseteq S_2$, then $S_2^{\perp} \subseteq S_1^{\perp}$. **Proof:** *Let us assume $S_1 \subseteq S_2$ and consider $x \in S_2^{\perp}$. For any $y \in S_1$, our hypothesis allows us to set that $y \in S_1$, and therefore $\Re(y^* x) = 0$ which means that $x \in S_1^{\perp}$.* $\square$

5. **Cauchy-Schwarz Inequality:** Given $u, v \in \mathbb{C}^n$, the Cauchy-Schwarz Inequality states:

$$\Re(u^* v) \leq \|u\| \|v\|.$$

Proof: *Let us first note that for any $x, y \in \mathbb{C}^n$:*

$$0 \leq \|x - y\|^2 = (x - y)^*(x - y) = x^* x + y^* y - x^* y - y^* x \|x\|^2 + \|y\|^2 - 2\Re(x^* y),$$

since $y^* x = \overline{x^* y}$. Therefore: $\Re(x^* y) \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2$. Now considering $x = \frac{u}{\|u\|}$ and $y = \frac{v}{\|v\|}$, one directly remarks that $\|x\| = \|y\| = 1$ and one then obtains $\Re(u^* v) \leq \|u\| \|v\|$. $\square$

3.3 Echelon decomposition

Given a matrix $A \in \mathcal{M}_{p,n}$, there exist invertible matrices $S_1, \ldots, S_t \in \mathcal{M}_p$ (these are products of elementary matrices $P_{i,j}$, $M_i(\lambda)$ or $G_{i,j}(\lambda)$) such that $C := S_t \cdots S_1 A$ is in echelon form, i.e., either $C = 0$ or

$$C = \left(\begin{array}{c|cc|cc|cc|c|cc}
 & 1 & * & 0 & * & 0 & * & & 0 & * \\ \cline{2-3}
 & & & 1 & * & 0 & * & & \vdots & \vdots \\ \cline{4-5}
 & & & & & 1 & \vdots & & \vdots & \vdots \\ \cline{6-7}
0 & & & & & & & & \vdots & \vdots \\
 & 0 & & & & & & \ddots & 0 & \vdots \\
 & & & 0 & & & & & 1 & * \\ \cline{9-10}
 & & & & & 0 & & & & \\
 & & & & & & & & 0 &
\end{array}\right)$$

Here $*$ denotes an arbitrary (zero or nonzero) little row of C. More precisely, $C = [C_{i,j}]$ is either the zero matrix, or there exists a sequence of natural numbers $j_1, \ldots, j_r$ (these are called the "steps" of the echelon form), where $1 \leq j_1 < \cdots < j_r \leq n$ and $1 \leq r \leq \min(n, p)$, such that

1. $c_{ij} = 0$ for $1 \leq i \leq r$ and $1 \leq j < j_i$,
2. $c_{ij} = 0$ for $r < i \leq p$ and $1 \leq j \leq n$,
3. $c_{i,j_i} = 1$ for $1 \leq i \leq r$ and all other entries in column j_i are zero.

If $n = p$, then A is invertible if and only if $C = I_n$. In this case $A^{-1} = S_t \cdots S_1$. In all cases, r is equal to the rank of A. (Proof not provided here).

3.4 Determinant

Let $AB \in \mathcal{M}_n(\mathbb{K})$, $\lambda \in \mathbb{K}$. The following properties hold:

1. $\det(A^T) = \det(A)$. **Proof:** *Note that the mapping $\sigma \in \mathfrak{S}_n \mapsto \sigma^{-1} \in \mathfrak{S}_n$ is a bijection of $\mathfrak{S}_n$ (it is even an injection), therefore one has the identity:*

$$\begin{aligned}
\det(A) &= \sum_{\sigma \in \mathfrak{S}_n} Sgn(\sigma) A_{1,\sigma(1)} \cdots A_{n,\sigma(n)} \\
&= \sum_{\sigma \in \mathfrak{S}_n} Sgn(\sigma^{-1}) A_{1,\sigma^{-1}(1)} \cdots A_{n,\sigma^{-1}(n)} = \sum_{\sigma \in \mathfrak{S}_n} Sgn(\sigma) A_{\sigma(1),1} \cdots A_{\sigma(n),n} = \det(A^T),
\end{aligned}$$

since $Sgn(\sigma^{-1}) = Sgn(\sigma)$. $\square$

2. The determinant is n-linear on the columns and on the rows of a matrix. Meaning that given a matrix $A = (a_1, \dots, a_n) \in \mathcal{M}_n(\mathbb{K})$, given $i \in [n]$, if $a_i = \alpha b_i + \beta c_i$, with $\alpha, \beta \in \mathbb{K}$ and $b_i, c_i \in \mathbb{K}^n$:

$$\det(A) = \alpha \det(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots a_n) + \beta \det(a_1, \dots, a_{i-1}, c_i, a_{i+1}, \dots a_n)$$

Proof: *The n-linearity on the columns is a direct consequence of the signature Formula of Leibniz. For the result on the rows, one can use Item 1.* $\square$

3. Given $i, j \in [n]$, $i \neq j$: $\det(P_{i,j}A) = \det(AP_{i,j}) = \det(P_{i,j})\det(A) = -\det(A)$. In particular we see that $\det(P_{i,j}) = -1$. **Proof:** *We assume $i < j$. The matrix $P_{i,j}A$ is equal to the matrix A with a swapping between the i^{th} and the j^{th} row, therefore the signature Formula of Leibniz gives us:*

$$\det(P_{i,j}A) = \sum_{\sigma \in \mathfrak{S}_n} Sgn(\sigma) A_{1,\sigma_1} \cdots A_{i-1,\sigma_{i-1}} A_{j,\sigma_i} A_{i+1,\sigma_{i+1}} \cdots A_{j-1,\sigma_{j-1}} A_{i,\sigma_j} A_{j+1,\sigma_{j+1}} \cdots A_{n,\sigma_n}.$$

Now, denoting $\tau \in \mathfrak{S}_n$, the transposition between i and j ($\tau(i) = j$, $\tau(j) = i$ and for all $k \in [n] \setminus \{i, j\}$, $\tau(k) = k$) we know that the mapping $\imath_\tau : \sigma \in \mathfrak{S}_n \mapsto \sigma \circ \tau \in \mathfrak{S}_n$ is an injection of $\mathfrak{S}_n$, therefore, one can replace σ with $\imath_\tau(\sigma)$ in the summand to obtain:

$$\det(P_{i,j}A) = \sum_{\sigma \in \mathfrak{S}_n} Sgn(\imath_\tau(\sigma)) A_{1,\sigma_1} \cdots A_{i-1,\sigma_{i-1}} A_{j,\sigma_j} A_{i+1,\sigma_{i+1}} \cdots A_{j-1,\sigma_{j-1}} A_{i,\sigma_i} A_{j+1,\sigma_{j+1}} \cdots A_{n,\sigma_n}.$$

Noting that $Sgn(\imath_\tau(\sigma)) = -Sgn(\sigma)$, one recognize the determinant of A and obtain the looked for identity. $\square$

4. 🕮 If A has two identical rows or two identical columns, then $\det A = 0$. **Proof:** *We show the result when A has two identical rows (of course the same result holds for the columns thanks to Item 1). Simply note that if the i^{th} and the j^{th} rows are identical with $i \neq j$, then $P_{i,j}A = A$ and Item 3 yields to $\det(A) = \det(P_{i,j}A) = -\det(A)$ which implies that $\det(A) = 0$.* $\square$

5. $\det(M_i(\lambda)A) = \det(AM_i(\lambda)) = \det(M_i(\lambda))\det(A) = \lambda\det(A)$. In particular we see that $\det(M_i(\lambda)) = \lambda$. **Proof:** *It is immediate:*

$$\det(M_i(\lambda)A) = \sum_{\sigma \in \mathfrak{S}_n} Sgn(\sigma) A_{1,\sigma_1} \cdots A_{i-1,\sigma_{i-1}} \lambda A_{i,\sigma_i} A_{i+1,\sigma_{i+1}} \cdots A_{n,\sigma_n} = \lambda \det(A).$$

$\square$

6. $\det(G_{i,j}(\lambda)A) = \det(AG_{i,j}(\lambda)) = \det(G_{i,j}(\lambda))\det(A) = \det(A)$. In particular $\det(G_{i,j}(\lambda)) = 1$. **Proof:** *The n-linearity of the determinant given in Item 2 provides:*

$$\det(AG_{i,j}(\lambda)) = \det(A) + \lambda \det(a_1, \dots, a_{i-1}, a_j, a_i, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n) = \det(A),$$

thanks to Item 4. $\square$

7. 🕮 Given a matrix $A \in \mathcal{M}_n$ having the triangular superior block decomposition:

$$A = \left(\begin{array}{c|c} B & C \\ \mathbf{0} & D \end{array}\right), \quad \text{with } B \in \mathcal{M}_{p,p},\ C \in \mathcal{M}_{p,q}, \text{ and } D \in \mathcal{M}_{q,q}, \text{ where } n = p + q,$$

then its determinant expresses:

$$\det A = \det(B)\det(D)$$

Proof: *Note that is we apply the Signature formula on A, all the $\sigma \in \mathfrak{S}_n$ such that $\sigma(\{p+1, \dots, n\}) \cap \{1, \dots, p\} \neq \emptyset$ will provide empty products in the summation, one can therefore merely sum on:*

$$\begin{aligned} \mathfrak{S}_{p,q} &\equiv \{\sigma \in \mathfrak{S}_{p+q} \mid \sigma(\{p+1, \dots, p+q\}) \subset \{p+1, \dots, p+q\}\} \\ &= \{\sigma \in \mathfrak{S}_{p+q} \mid \sigma(\{1, \dots, p\}) = \{1, \dots, p\} \textit{ and } \sigma(\{p+1, \dots, p+q\}) = \{p+1, \dots, p+q\}\} \end{aligned}$$

Then it is quite easy to see that this set is in bijection with[3] $\mathfrak{S}_{[p]} \times \mathfrak{S}_{p+[q]}$ *through the map:*

$$\begin{array}{rcccl} \Phi: & \mathfrak{S}_{[p]} \times \mathfrak{S}_{p+[q]} & \longrightarrow & \mathfrak{S}_{p,q} \\ & (\sigma,\theta) & \longmapsto & i \mapsto \begin{cases} \sigma(i) \text{ if } i \leq p \\ \theta(i) \text{ if } i \geq p+1. \end{cases} \end{array}$$

The Signature formula of Leibniz then gives us:

$$\begin{aligned} \det(A) &= \sum_{\sigma \in \mathfrak{S}_{p,q}} Sgn(\sigma) A_{1,\sigma(1)} \cdots A_{n,\sigma(n)} \\ &= \sum_{\sigma \in \mathfrak{S}_{[p]}} \sum_{\theta \in \mathfrak{S}_{p+[q]}} Sgn(\Phi(\sigma,\theta)) A_{1,\sigma(1)} \cdots A_{p,\sigma(p)} A_{p+1,\theta(p+1)} \cdots A_{n,\theta(n)} \; = \; \det(B)\det(D), \end{aligned}$$

since $Sgn(\Phi(\sigma,\theta)) = Sgn(\sigma)Sgn(\theta)$ □

8. The determinant of an $n \times n$ matrix A can be expressed using the formula involving minors:

$$\forall j \in [n]: \quad \det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{i,j} |A_{-i,j}| \quad \text{and} \quad \forall i \in [n]: \quad \det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{i,j} |A_{-i,j}|$$

where $|A_{-i,j}|$ is the (i,j) minor of A.
Proof: *Employing Item 2 let us develop the determinant around the* j^{th} *column* $a_j = \sum_{i=1}^{n} A_{i,j} e_i$*:*

$$\det(A) = \sum_{i=1}^{n} A_{i,j} \det(a_1, \ldots, a_{j-1}, e_i, a_{j+1}, \ldots, a_n).$$

Now, recalling that $A_{-i,j}$ *is the matrix* A *after removing the* i^{th} *row and the* j^{th}*, we naturally denote:*

$$\hat{A}_{-i,j} = (a_1, \ldots, a_{j-1}, e_i, a_{j+1}, \ldots, a_n). \tag{1.1}$$

It satisfies:

$$P_{1,2} \cdots P_{i-1,i} \hat{A}_{-i,j} P_{j-1,j} \cdots P_{1,2} = \left(\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & A_{-i,j} \\ 0 & \end{array} \right).$$

Therefore Items 3 and 7 allow us to set that $\det(\hat{A}_{-i,j}) = (-1)^{j-1}(-1)^{j-1} \cdot 1 \cdot |A_{-i,j}| = (-1)^{i+j}|A_{-i,j}|$, *which implies the formula. The formula developed on the* i^{th} *row is deduced thanks to Item 1.* □

9. 🕮 $\det(AB) = \det(A) \cdot \det(B)$. **Proof:** *According to Subsection 3.3, we can find invertible elementary matrices* $S_1, \ldots, S_t$ *such that* $C = S_t \cdots S_1 A$ *is in its echelon form. Invoking Items 3, 5 and 6, the determinant of* A *can be expressed as the product of the determinants of these matrices and* C*, i.e.,*

$$\det(A) = \det(S_1^{-1}) \cdots \det(S_t^{-1}) \det(C),$$

and similarly for the product of A *and any matrix* B*,*

$$\det(AB) = \det(S_1^{-1} \cdots S_t^{-1} CB) = \det(S_1^{-1}) \cdots \det(S_t^{-1}) \det(CB).$$

We consider two scenarios. If A *is non-invertible, then* C *and consequently* CB *must contain a zero row, leading to* $\det(C) = \det(CB) = 0$*. This implies* $\det(A) = 0$*, and therefore* $\det(AB) = \det(A)\det(B) = 0$*. Conversely, if* A *is invertible, then* C *must be the identity matrix* I_n*, due to its echelon form. It follows that* $\det(I_n) = 1$*, and hence* $\det(AB) = \det(A)\det(B)$*.* □

10. 🕮🖝 If A is invertible, $\det A \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$. **Proof:** *Item 9 and the identity* $1 = \det(I_n)$ *imply* $1 = \det(AA^{-1}) = \det(A)\det(A^{-1})$*, which gives us* $\det(A^{-1}) = \frac{1}{\det(A)}$*.* □

11. 🕮If $A \in \mathcal{M}_2$: $\det(A) = A_{1,1}A_{2,2} - A_{1,2}A_{2,1}$.

[3] Where we defined $p + [q] \equiv \{p+i, i \in [q]\} = \{p+1, \ldots, p+q\}$

3.5 Inverse/Transpose of a Matrix

Let $A, B \in \mathcal{M}_n$ be two invertible matrices and $C \in \mathcal{M}_{p,m}$ and $D \in \mathcal{M}_{m,n}$. The following properties hold:

1. $(CD)^T = D^T C^T$ and $(CD)^* = D^* C^*$. **Proof:** *It is simply deduced from the matricial product. If one denotes $M = D^T C^T \in \mathcal{M}_{n,p}$ and considers $i \in [n]$ and $j \in [p]$: $M_{i,j} = \sum_{k=1}^{m} D_{k,i} C_{j,k} = (CD)_{j,i}$. The hermitian transpose is simply deduced bby the distribution of the complex conjugate on the product and on the sum (Given $z_1, z_2 \in \mathbb{C}$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ and $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$).* □

2. A square matrix is injective iif it is surjective. **Proof:** *It is a consequence of the Dimension Theorem: $\dim(Ker(A)) + \dim(Im(A)) = n$ therefore:*

$$A \text{ surjective} \iff \dim(Im(A)) = n \iff \dim(Ker(A)) = n \iff A \text{ injective.}$$

□

3. A matrix is invertible iif it is injective or surjective (which is equivalent to being injective AND surjective). **Proof:** *If A is invertible, then $Ax = 0 \implies x = A^{-1}0 = 0$ thus $Ker(A) = \{0\}$ and A is injective. If we assume that A is injective and surjective then the echelon decomposition of A given in Subsection 3.3 ensures the existence of an invertible matrix S such that $A = SC$ and C has an echelon form. Given $x \in Ker(C)$, we know that $Ax = SCx = 0$ thus $x \in Ker(A) = \{0\}$ and therefore C is injective as A. Now since C is squared, the previous item allows to set that C is also surjective, and therefore, the only echelon form possible for C is without the first columns of zeros (by injectivity) and without the last rows of zeros (by surjectivity). Finally C must have the form:*

$$C = \begin{pmatrix}
1 & * & 0 & * & 0 & * & & 0 & * \\
 & & 1 & * & 0 & * & & \vdots & \vdots \\
 & & & & 1 & \vdots & & \vdots & \vdots \\
 & & & & & & & \vdots & \vdots \\
 & 0 & & & & & \ddots & 0 & \vdots \\
 & & & 0 & & & & 1 & *
\end{pmatrix}.$$

Again, since the matrix C is a square matrix, all the "$*$" little rows must have zero length and consequently $C = I_n$ which finally ensures that $A = S$ is invertible. □

4. A is invertible iif its columns and its rows form a basis of $\mathbb{K}^n$. **Proof:** *We show the result for the columns of $A = (a_1, \dots, a_n)$. We assume that A is invertible and we consider n scalars $\alpha_1, \dots, \alpha_n$ such that $\alpha_1 a_1 + \dots + \alpha_n a_n = 0$. This equation write matricially as $A\alpha = 0$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$, multiplying by A^{-1} one the right, we see that $\alpha = 0$, therefore $a_1, \dots, a_n$ are linearly independent and one can conclude with Subsection 3.1, Item 5. If we assume now that $a_1, \dots, a_n$ are linearly independent, then the equation $A\alpha = 0$ implies $\alpha = 0$ which exactly means that A is injective and one can conclude with Item 3* □

5. When $p = m = n$, if $CD = I_n$ or $DC = I_n$ then C and D are invertible and $D = C^{-1}$. **Proof:** *If $DC = I_n$ then $\{0\} \subset Ker(C) \subset Ker(DC) = Ker(I_n) = \{0\}$ and therefore $Ker(C) = \{0\}$ which implies that C is injective and therefore invertible (see Item 3). Multiplying by C^{-1} on the left, one obtains $D = C^{-1}$.*
If $CD = I_n$ then $\mathbb{K}^n \supset Im(C) \supset Im(CD) = Im(I_n) = \mathbb{K}^n$ which implies that $Im(C) = \mathbb{K}^n$ thus C is surjective and therefore invertible. □

6. $(A^{-1})^{-1} = A$. **Proof:** *$A^{-1}A = I_n$, thus Item 5 allows to set that $A = (A^{-1})^{-1}$.* □

7. $(AB)^{-1} = B^{-1}A^{-1}$. **Proof:** *$B^{-1}A^{-1}AB = B^{-1}B = I_n$, we then conclude with Item 5.* □

8. Given a scalar k: $(kA)^{-1} = \frac{1}{k}A^{-1}$ if $k \neq 0$.

9. $(A^T)^{-1} = (A^{-1})^T$ and $(A^*)^{-1} = (A^{-1})^*$. **Proof:** *Let us simply note thanks to Item 1 that:* $(A^{-1})^T A^T = (AA^{-1})^T = I_n$ *and conclude with Item 5, the same argument works for the conjugate transpose.* □

10. Given $M \in \mathcal{M}_n$ and denoting $C = \mathrm{Com}(M)$, the comatrix of M: $MC^T = C^T M = \det(M) I_n$.

    Proof: *Recall that* $C \in \mathcal{M}_n$ *is defined by* $C_{i,j} = (-1)^{i+j}|M_{-i,j}|$. *Then the matrix* $D = C^T M$ *satisfies:*

    $$D_{i,j} = \sum_{k=1}^n C_{k,i} M_{k,j} = \sum_{k=1}^n (-1)^{i+k} |M_{-k,i}| M_{k,j}.$$

    Consider m_ℓ *to be the* ℓ*-th column of* M*, and define as in* (1.1)*, the "hat" matrix:*

    $$\hat{M}_{-k,i} = [m_1, \dots, m_{i-1}, e_k, m_{i+1}, \dots, m_n] \in \mathbb{R}^{n,n}.$$

    We saw in the proof of Subsection 3.4, Item 8 that $\det(M_{-k,i}) = (-1)^{k+i} \det(\hat{M}_{-k,i})$. *Now, considering the determinant's linearity with respect to columns, we obtain*

    $$D_{i,j} = \sum_{k=1}^n (-1)^{i+k}(-1)^{k+i} M_{kj} |\hat{M}_{-k,i}| = \det(m_1, \dots, m_{i-1}, m_j, m_{i+1}, \dots, m_n) = \begin{cases} 0, & i \neq j, \\ \det(M), & i = j, \end{cases}$$

    thanks to Subsection 3.4, Item 4. One finally obtains $D_{i,j} = \delta_{i,j} \det(M)$*, and thus* $C^T M = \det(M) I_n$. *A similar argument relying on the determinant's linearity with respect to rows shows that* $MC^T = \det(M) I_n$. □

11. Given four scalars $a, b, c, d \in \mathbb{K}$ such that $ad - cb \neq 0$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ **Proof:** *Consequence of Item 10.* □

12. Given a matrix $A \in \mathcal{M}_n(K)$ and a basis $\mathcal{B} = (w_1, \dots, w_n)$ of $\mathbb{K}^n$, we denote $P \equiv ([w_1]_\mathcal{B}, \dots, [w_n]_\mathcal{B}) \in \mathcal{M}_n$, the change of basis matrix to $\mathcal{B}$. Given a vector $x \in \mathbb{K}^n$, $[x]_\mathcal{B} = P^{-1}x$ (denoting $y \equiv [x]_\mathcal{B}$, one has the identity $x = Py$).

    The representation $[A]_\mathcal{B}$ of A in $\mathcal{B}$ is defined as:

    $$[A]_\mathcal{B} = ([Aw_1]_\mathcal{B}, \dots, [Aw_n]_\mathcal{B}).$$

    it satisfies $[A]_\mathcal{B} = P^{-1}AP$ and:

    $$\forall x \in \mathbb{K}^n : \quad [Ax]_\mathcal{B} = [A]_\mathcal{B}[x]_\mathcal{B}.$$

    Proof: *Let us simply express:*

    $$x = y_1 w_1 + \cdots + y_n w_n = Py.$$

    To prove the second result, let us express from last result:

    $$P^{-1}AP = P^{-1}(Aw_1, \dots, Aw_n) = ([Aw_1]_\mathcal{B}, \dots, [Aw_n]_\mathcal{B}).$$

    Finally, given $x \in \mathbb{K}^n$*, one has naturally:*

    $$[Ax]_\mathcal{B} = P^{-1}Ax = P^{-1}APP^{-1}x = [A]_\mathcal{B}[x]_\mathcal{B}.$$

    □

3.6 Eigenvalues

Let us consider a square matrix $A \in \mathcal{M}_n$ with k distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. We denote $E_{\lambda_1}, \ldots, E_{\lambda_k}$ the respective eigenspaces and $\alpha_1, \ldots, \alpha_k$ the respective geometric multiplicities (i.e the dimension of $E_{\lambda_1}, \ldots, E_{\lambda_k}$).

1. $E_{\lambda_1} + \cdots + E_{\lambda_k}$ is a direct sum[4].
   Proof: *For $r \in [k]$, let us assume that $r = 1$ or $E_{\lambda_1} + \cdots + E_{\lambda_r}$ is in direct sum and consider $x_1 \in E_{\lambda_1}, \ldots, x_{r+1} \in E_{\lambda_{r+1}}$ all different from zero and $\alpha_1, \ldots, \alpha_{r+1}$ such that $\alpha_1 x_1 + \cdots + \alpha_{r+1} x_{r+1} = 0$, then applying A on the left, one obtains:*

   $$\alpha_1 \lambda_1 x_1 + \cdots + \alpha_{r+1} \lambda_{r+1} x_{r+1} = 0$$

   subtracting this equation with $\lambda_{r+1} \cdot (\alpha_1 x_1 + \cdots + \alpha_{r+1} x_{r+1} = 0)$, one obtains:

   $$\alpha_1(\lambda_1 - \lambda_{r+1}) x_1 + \cdots + \alpha_r(\lambda_r - \lambda_{r+1}) x_r = 0,$$

   which implies (since $\lambda_1, \ldots, \lambda_r$ are all different from λ_{r+1} and $x_1, \ldots, x_r$ are free by iteration hypothesis): $\alpha_1 = \cdots = \alpha_r$. Then α_{r+1} also cancels (since $x_{r+1} \neq 0$) and we deduce that $E_{\lambda_1} + \cdots + E_{\lambda_{r+1}}$ is in direct sum. $\square$

2. If A has n distinct eigenvalues then it is diagonalizable (the converse is not true).
   Proof: *If A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ ($n = k$), then Item 1 allows us to set that $\mathbb{K}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n}$. Considering $x_i \in E_{\lambda_i}$, we know that $x_1, \ldots, x_n$ is linearly independent and therefore we know from Subsection 3.1, Item 5 that it is a basis of $\mathbb{K}^n$. Writing A in this basis (see Subsection 3.5, Item 12) gives a diagonal matrix Diag($\lambda_1, \ldots, \lambda_n$). If A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, we know that:*

   - *For $i \in [n]$, $\dim E_{\lambda_i} > 1$*
   - *$E_{\lambda_1}, \ldots, E_{\lambda_n}$ are in direct sum.*

   Now:

   $$\dim(E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n}) = \dim E_{\lambda_1} + \cdots + \dim E_{\lambda_n} \geq n = \dim(\mathcal{K}^n) \quad \textit{and} \quad E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n} \subset K^n,$$

   thus $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n} = K^n$ and subsequently $\dim E_{\lambda_1} = \cdots = \dim E_{\lambda_n} = 1$. For all $k \in [n]$, let us pick one non zero vector in E_k such that $E_k = \mathbb{K} v_k$. Then one can easily show from Item 1 ($E_{\lambda_1}, \ldots, E_{\lambda_n}$ in direct sum) that $v_1, \ldots, v_n$ is free:

   $$(a_n v_1 + \cdots + a_n v_n = 0) \quad \Rightarrow \quad a_1 v_1 = \cdots = a_n v_n = 0 \quad \Rightarrow \quad a_1 = \cdots = a_n = 0,$$

   since the vectors $v_1, \ldots, v_n$ are all different from zero. Being a free family of n elements in a space of dimension n, we know from Subsection 3.1, Item 5 that $\mathcal{B} \equiv (v_1, \ldots, v_n)$ is a basis of $\mathcal{K}^n$. Then $\forall k \in [n]$, $v_k \in E_{\lambda_k}$, and therefore: $A v_k = \lambda_k v_k = 0 v_1 + \cdots + \lambda_k v_k + \cdots + 0 v_n$, that exactly means that $[A v_k]_{\mathcal{B}} = \lambda_k e_k$ (0 everywhere and λ_k in the i^{th} entry). One can then express:

   $$[A]_{\mathcal{B}} = ([A v_n]_{\mathcal{B}}, \ldots, [A v_n]_{\mathcal{B}}) = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix},$$

   the matrix A is diagonalizable. $\square$

3. A is diagonalizable iif $\sum_{i=1}^k \alpha_i = n$.
   Proof: *Let us first note that the identity $\sum_{i=1}^k \alpha_i = n$ is equivalent to $\mathbb{K}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ thanks to Item 1 combined with Subsection 3.1, Items 7 and 8. Then 'if' part is proven the same way as Item 2 but this time, one needs to consider in each eigenspace E_{λ_i} a basis of α_i elements, that, put together, will allow to diagonalize A.*

[4] Recall that it means that if there exist $x_1 \in E_{\lambda_1}, \ldots, x_k \in E_{\lambda_k}$ such that $x_1 + \cdots + x_k = 0$ then $x_1 = \cdots = x_k = 0$

To show the "only if" part, let us assume A is diagonalizable and introduce the basis $\mathcal{B} = (u_1 \ \cdots \ u_n)$ that diagonalizes A. We know that there exists k parameters $\gamma_1, \ldots \gamma_k \in \mathbb{N}$ such that the representation of A is $\mathcal{B}$ is:

$$[A]_{\mathcal{B}} = P^{-1}AP = \begin{pmatrix} \boxed{\lambda_1 I_{\gamma_1}} & & & (0) \\ & \boxed{\lambda_2 I_{\gamma_2}} & & \\ & & \ddots & \\ (0) & & & \boxed{\lambda_k I_{\gamma_k}} \end{pmatrix},$$

we know that $\sum_{i=1}^k \gamma_i = k$, our goal is then to show that $\forall i \in [k]$ $\gamma_i = \alpha_i$ (the algebraic multiplicity $\dim(E_{\lambda_i})$ associated to λ_i). For this let us show that:

$$E_{\lambda_i} = \mathbb{K}u_{\alpha_1+\cdots+\alpha_{i-1}+1} + \cdots + \mathbb{K}u_{\alpha_1+\cdots+\alpha_i}.$$

This is done through the introduction of the change of basis matrix $P = (u_1, \ldots, u_n) \in \mathcal{M}_n$ and equivalence sequence:

$$\begin{aligned} x \in E_{\lambda_i} &\iff Ax - \lambda_i x = 0 \iff P^{-1}(A - \lambda_i I_n)PP^{-1}x = 0 \iff ([A]_{\mathcal{B}} - \lambda_i I_n)P^{-1}x = 0 \\ &\iff P^{-1}x \in \mathbb{K}e_{\alpha_1+\cdots+\alpha_{i-1}+1} \oplus \cdots \oplus \mathbb{K}e_{\alpha_1+\cdots+\alpha_i} \iff x \in \mathbb{K}u_{\alpha_1+\cdots+\alpha_{i-1}+1} \oplus \cdots \oplus \mathbb{K}u_{\alpha_1+\cdots+\alpha_i}, \end{aligned}$$

since $\forall i \in [n]$, $Pe_i = u_i$ and:

$$[A]_{\mathcal{B}} - \lambda_i I_n = \begin{pmatrix} \boxed{(\lambda_1 - \lambda_i)I_{\gamma_1}} & & & & (0) \\ & \ddots & & & \\ & & \boxed{0 I_{\gamma_i}} & & \\ & & & \ddots & \\ (0) & & & & \boxed{(\lambda_k - \lambda_i)I_{\gamma_k}} \end{pmatrix}.$$

One can then directly deduce that $\alpha_i = \dim(E_{\lambda_i}) = \gamma_i$ which allows us to conclude. □

4. 🕮 A is diagonalizable if and only if:

$$\mathbb{K}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k},$$

(in other words, $\mathbb{K}^n$ has a basis of eigenvectors for A). **Proof:** *As before, consequence of Subsection 3.1, Item 8.* □

5. 🕮 When A is diagonalizable, the determinant of A is equal to the product of its eigenvalues:

$$\det(A) = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_k^{\alpha_k}.$$

6. 🕮 When A is diagonalizable, the trace of A is equal to the sum of its eigenvalues:

$$\operatorname{tr}(A) = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \cdots + \alpha_k\lambda_k.$$

3.7 Polynomials

1. 🕮 A matrix $A \in \mathcal{M}_{p,n}$ commutes with any $P(A)$ where P is a polynomial of $\mathbb{C}[X]$ and consequently polynomials of A mutually commute.

2. **Euclidean Division.** Given two polynomial $A, B \in \mathbb{K}[X]$ such that $B \neq 0$, there exist two polynomials Q, R such that:

   - $A = BQ + R$
   - $\deg(R) < \deg(B)$

(Proof not provided here).

3. **Bezout's theorem.** If Q is the greatest common divisor of $n \in \mathbb{N}$ polynomials $P_1, \dots, P_n$ (all different from zero), then there exist n polynomials $U_1, \dots, U_n$ such that:

$$P_1 U_1 + \cdots + P_n U_n = Q$$

Proof: *The set of monic[5] polynomials $D \in \mathbb{K}[X]$ such that there exists n polynomials $U_1, \dots, U_n \in \mathbb{K}[X]$ satisfying:*

$$P_1 U_1 + \cdots + P_n U_n = D \tag{1.2}$$

is non empty (any of $P_1, \dots, P_n$ with a correct scalar factorization belong to this set), therefore one can consider the polynomial D with smallest degree satisfying this identity (since D is monic, $D \neq 0$). Given $i \in [n]$ let's perform the euclidean division of P_i with D. There exist $Q_i, R_i \in \mathbb{K}[X]$ such that $P_i = DQ_i + R_i$ and $\deg(R_i) < \deg(D)$. Now, if one replaces D in the euclidean division by its value given by identity (1.2), one obtains:

$$P_i = (P_1 U_1 + \cdots + P_n U_n)\, Q_i + R_i,$$

Therefore:

$$-P_1 U_1 Q_i - \cdots - P_{i-1} U_{i-1} Q_i + P_i(1 - U_i Q_i) - \cdots - P_n U_n Q_i = R_i.$$

Now by minimality of D and since $\deg R_i < \deg D$, one can deduce that $\forall i \in [n] : R_i = 0$, and therefore D divides each $P_1, \dots, P_n$. To show that D is the greatest common divisor, let us simply note that if a polynomial $\Delta \in \mathbb{K}[X]$ divides each $P_1, \dots, P_n$ then it also divides D thanks to (1.2). □

4. 🕮 **Fundamental Theorem of Algebra.** Every non-constant polynomial of $\mathbb{C}[X]$ has a root in $\mathbb{C}$. (Proof not provided here)

5. 🕮☛ Given $P \in \mathbb{C}[X]$ and $A \in \mathcal{M}_n$, if $P(A) = \mathbf{0}$, then for any λ, eigenvalue of A, $P(\lambda) = 0$. **Proof:** *Assuming that $P(A) = \mathbf{0}$ and considering an eigenvector x associated to the eigenvalue λ we have the identity $0 = P(A)x = P(\lambda)x$.* □

3.8 Characteristic polynomial

We consider below a matrix $A \in \mathcal{M}_n(\mathbb{K})$ and denote χ_A its characteristic polynomial. Be careful, geometric (dimension of eigenspaces) and algebraic (monimial degrees in χ_A) multiplicities can be completely different! For instance $A = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ has just one eigenvalue: $\mathrm{Sp}(A) = \{1\}$. Now the algebraic multiplicity associated to this eigenvalue is 2 ($\chi_A = (X-1)^2$) when the geometric multiplicity is 1 ($E_1 = \mathbb{R}e_1$). Note that $\chi_A = \chi_{I_2}$ but $A \neq I_2$ which means that the characteristic polynomial actually does not fully characterize a matrix.

1. 🕮☛ The characteristic polynomial of A is the same as the characteristic polynomial of the representation of A in any basis of $\mathbb{K}^n$. **Proof:** *Simply note from Subsection 3.5, item 12 that given a basis $\mathcal{B}$ of $\mathbb{K}^n$:*

$$\chi_{[A]_{\mathcal{B}}} = \chi_{P^{-1}AP} = \det(XI_n - P^{-1}AP) = \det(P^{-1}(XI_n - A)P) = \det(XI_n - A) = \chi_A.$$

□

2. 🕮 The set of roots of χ_A coincides with $\mathrm{Sp}(A)$ and χ_A is of degree n. **Proof:** *It is simply a consequence of the definition of the characteristic polynomial: $\chi_A = \det(XI_n - A)$ therefore $\chi_A(\lambda) = 0 \Leftrightarrow A - \lambda I_n$ singular $\Leftrightarrow \lambda$ eigenvalue of A.* □

3. 🕮 If χ_A has an elementary factor then A has at least one eigenvalue. In particular thanks to Subsection 3.7, Item 4, any matrix of $\mathcal{M}_n(\mathbb{C})$ has one complex eigenvalue.

[5] Recall that a monic polynomial is a polynomial whose coefficient associated to the highest degree monomial is equal to one. Ex: $X^2 + 6$ is monic, $2X - 1$ is not.

4. Trace and determinant appear in the expression of the characteristic polynomial $\chi_A = a_n X + a_{n-1}X^{n-1} + \cdots + a_0$:

$$a_n = 1, \qquad a_{n-1} = -\mathrm{Tr}(A) \qquad \text{and} \qquad a_0 = (-1)^n \det(A).$$

In particular when $n = 2$: $\chi_A = X^2 - \mathrm{Tr} A X + A^2$. **Proof:** *Let us express the characteristic polynomial with the signature formula of Leibniz:*

$$\chi_A = \det(XI_n - A) = \sum_{\sigma \in \mathfrak{S}_{p,q}} Sgn(\sigma)(X\delta_{1,\sigma(1)} - A_{1,\sigma(1)}) \cdots (X\delta_{n,\sigma(n)} - A_{n,\sigma(n)}).$$

One sees immediately from this formula that χ_A *is of degree* n, $a_n = \det(I_n) = 1$, $a_0 = \det(-A) = (-1)^{-n}\det(A)$ *and monomials of degree* $n-1$ *are only obtained in the sum for* $\sigma = Id_{\mathfrak{S}_n}$, *then* $Sgn(\sigma) = 1$ *and developing the product, one obtains* $a_{n-1} = -A_{1,1} - \cdots - A_{n,n} = -Tr(A)$. ☐

5. algebraic multiplicity $\geq$ geometric multiplicity. **Proof:** *The proof is similar to the proof of Subsection 3.6 Item 4. Considering a matrix* $A \in \mathcal{M}_{p,n}$, *one just needs to compute the characteristic polynomial of* A *expressed in an extension of a basis compatible with the direct sum of the eigenspaces* $\mathcal{B}'$*:*

$$[A]_{\mathcal{B}'} = \left(\begin{array}{cccc|c} \boxed{\lambda_1 I_{\alpha_1}} & & (0) & & * \\ & \boxed{\lambda_2 I_{\alpha_2}} & & & * \\ & & \ddots & & * \\ & (0) & & \boxed{\lambda_k I_{\alpha_k}} & * \\ & & & & R \end{array}\right),$$

Then, thanks to Subsection 3.4, Item 7, one can express: $\chi_A = \chi_{[A]_{\mathcal{B}'}} = (X - \lambda_1)^{\alpha_1} \cdots (X - \lambda_k)^{\alpha_k} \chi_C$, *we see that the algebraic multiplicities are all bigger than the algebraic multiplicities.* ☐

Lecture 2

Complexity of matrix computations

Complexity of an algorithm is measured by the float-point arithmetic operations, such as addition and multiplication, and division. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{n\times m}$, $\mathbf{B} \in \mathbb{R}^{m\times p}$:

- $\mathbf{x}^T\mathbf{y}$ requires m multiplications and $m-1$ additions: complexity of order $O(m)$.
- $\mathbf{Ax}$ requires $n(2m-1)$ additions and multiplications: complexity of order $O(nm)$.
- $\mathbf{AB}$ requires $p \cdot n(2m-1)$ additions and multiplications: complexity of order $O(pnm)$.

Complexity of Gram-Schmidt Procedure

Given a linearly independent set $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^m$, the Gram-Schmidt procedure is as follows:

For $i = 1, 2, \ldots, n$:

1. $\mathbf{q}_i = \mathbf{a}_i - \sum_{j=1}^{i-1}(\mathbf{q}_j^T\mathbf{a}_i)\mathbf{q}_j$.
2. Normalize $\mathbf{q}_i$ to obtain $\mathbf{q}_i = \frac{\mathbf{q}_i}{\|\mathbf{q}_i\|_2}$.

Output $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ as an orthonormal set.

For each iteration i:

- Every $\mathbf{q}_j^T\mathbf{a}_i$ takes $O(m)$.
- Computing $\mathbf{q}_i$ takes $(i-1)O(m) + O(m)$.
- Computing $\|\mathbf{q}_i\|_2$ takes $2O(m)$.

Therefore,

$$(i-1)O(m) + O(m) + 2O(m) = (i+2)O(m) = O(im)$$

.

Thus, the total complexity is

$$\sum_{i=1}^{n} O(im) = O\left(\sum_{i=1}^{n} im\right) = O\left(\frac{n(n+1)}{2}m\right) = O(n^2m)$$

.

Matrix multiplication Complexity

The Strassen algorithm, published by Volker Strassen in 1969, was a groundbreaking method for matrix multiplication that demonstrated that the general matrix multiplication algorithm was not optimal. It reduced the multiplication operations from 8 to 7 for a 2x2 matrix, thereby reducing the asymptotic complexity for larger matrices.

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \times \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix}$$

Standard algorithm

$$\begin{aligned} h_1 &= a_{1,1}b_{1,1} \\ h_2 &= a_{1,1}b_{1,2} \\ h_3 &= a_{1,2}b_{2,1} \\ h_4 &= a_{1,2}b_{2,2} \\ h_5 &= a_{2,1}b_{1,1} \\ h_6 &= a_{2,1}b_{1,2} \\ h_7 &= a_{2,2}b_{2,1} \\ h_8 &= a_{2,2}b_{2,2} \end{aligned}$$

$$\begin{aligned} c_{1,1} &= h_1 + h_3 \\ c_{1,2} &= h_2 + h_4 \\ c_{2,1} &= h_5 + h_7 \\ c_{2,2} &= h_6 + h_8 \end{aligned}$$

Strassen's algorithm

$$\begin{aligned} h_1 &= (a_{1,1} + a_{2,2})(b_{1,1} + b_{2,2}) \\ h_2 &= (a_{2,1} + a_{2,2})b_{1,1} \\ h_3 &= a_{1,1}(b_{1,2} - b_{2,2}) \\ h_4 &= a_{2,2}(b_{2,1} - b_{1,1}) \\ h_5 &= (a_{1,1} + a_{1,2})b_{2,2} \\ h_6 &= (-a_{1,1} + a_{2,1})(b_{1,1} + b_{1,2}) \\ h_7 &= (a_{1,2} - a_{2,2})(b_{2,1} + b_{2,2}) \end{aligned}$$

$$\begin{aligned} c_{1,1} &= h_1 + h_4 - h_5 + h_7 \\ c_{1,2} &= h_3 + h_5 \\ c_{2,1} &= h_2 + h_4 \\ c_{2,2} &= h_1 - h_2 + h_3 + h_6 \end{aligned}$$

- The Strassen algorithm achieves a reduction in the complexity of matrix multiplication through a divide-and-conquer strategy that recursively breaks down each matrix into four submatrices. For matrices of size $N = 2^n$, the complexity can be expressed using the recursive relation $f(n) = 7f(n-1) + \Theta(4^n)$, where $\Theta(4^n)$ represents the operations for the added and subtracted matrix combinations. The asymptotic complexity thus becomes $O(N^{\log_2 7}) = O(N^{2.8074}) \ll O(N^3)$.
- **Practical limitations:** not efficient for small matrices due to the overhead of additional additions and memory requirements. Plus somewhat reduced numerical stability. The algorithm is typically used for large matrices (500×500) where the trade-offs are justified by the performance gains.
- There exist theoretical improvement like the Coppersmith–Winograd algorithm and its optimized version that present a complexity of order $O(N^{2.37\cdots})$. However the constant in the big O is overwhelming and as a consequence these algorithms are useless for the range of matrices that can be handled on today computers.
- Matrix multiplication multiplicity admits $O(N^2)$ as a lower bound because any exact multiplication algorithm should at least make operation with the $2N^2$ entries of the two matrices.

Lecture 3

Polynomial characterization of Triangularizable and Diagonalizable matrices.

Theorem 3.1 (Schur). *A matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{C})$ (resp. $\mathcal{A} \in \mathcal{M}_n(\mathbb{R})$) is triangularizable with a unitary matrix*[1] *iif its characteristic polynomial χ_A can be split into elementary factors (i.e. of degree 1). In particular*[2] *any matrix is triangularizible in $\mathcal{M}_n(\mathbb{C})$ (thanks to Lecture 1, Subsection 3.7, Item 4).*

Proof. The "only if" part is obvious thanks to the definition of the Characteristic polynomial and Lecture 1, Subsection 3.4, Item 7.

The "if" part relies on an algorithmic proof which involves a sequence of reductions. Considering one root λ_1 of χ_A (it exists thanks to Lecture 1, Subsection 3.8, Item 3) x_1 is a normalized eigenvector of matrix A corresponding to the eigenvalue λ_1. We can extend the nonzero vector x_1 to form a basis $\{x_1, y_2, y_3, \ldots, y_n\}$ of $\mathbb{C}^n$.

Applying the Gram-Schmidt process, we obtain an orthonormal basis $\{x_1, z_2, \ldots, z_n\}$. These orthonormal vectors, arranged as columns, construct a unitary matrix U_1. The product $U_1^* A U_1$ reveals a matrix with the form

$$U_1^* A U_1 = \begin{bmatrix} \lambda_1 & * \\ 0 & A_1 \end{bmatrix},$$

where $A_1 \in M_{n-1}$ and has eigenvalues $\lambda_2, \ldots, \lambda_n$.

Note now from Lecture 1, Subsection 3.4, Item 7 that χ_{A_1} is a factor of χ_A, therefore one can repeat the above process: consider an eigenvalue λ_2 of $\mathcal{A}_2$, an associated normalized eigenvector $x_2 \in \mathbb{C}^{n-1}$ and find a unitary matrix $U_2 \in M_{n-1}$ such that

$$U_2^* A_1 U_2 = \begin{bmatrix} \lambda_2 & * \\ 0 & A_2 \end{bmatrix}.$$

Define $V_2 = \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix}$.

Both V_2 and $U_1 V_2$ are unitary, and $V_2^* U_1^* A U_1 V_2$ has the form

$$\begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & A_2 \end{bmatrix}.$$

We continue this process to generate unitary matrices $U_i \in M_{n-i+1}$, $i = 1, \ldots, n-1$ and unitary matrices $V_i \in M_n$, $i = 2, \ldots, n-1$. As a result, the matrix $U = U_1 V_2 V_3 \cdots V_{n-1}$ is unitary and $U^* A U$ yields a matrix in the desired upper triangular form.

[1] It means that there exists a unitary matrix U such that $U^* A U$ is triangular

[2] Because any polynomial of $\mathbb{C}[X]$ can be split into elementary factors, one says that $\mathbb{C}$ is "algebraically closed".

If all eigenvalues of $A \in M_n(\mathbb{R})$ are real, then corresponding eigenvectors can also be chosen to be real. Thus, the aforementioned steps can be performed using real arithmetic. $\square$

Remark 3.2. *The following matrix is not triangularizable in $\mathcal{M}_n(\mathbb{R})$ since its characteristic polynomial does not admit elementary factors:*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

indeed $\chi_A = X^2 + 1$.

Definition 3.1 (Dense sets)**.** *Given a set V endowed with a metric d, we say that a subset $S \subset V$ is dense in V if for all $x \in V$, $\varepsilon > 0$ there exists $y \in S$ such that:*

$$d(x, y) \leq \varepsilon.$$

On $\mathcal{M}_{p,n}$, a natural metric can be introduced thanks to the Frobenius norm (or Hilbert-Shmidt norm) defined for any $A \in \mathcal{M}_{p,n}$ as:

$$\|A\|_F = \sqrt{\operatorname{Tr}(A^* A)}.$$

Corollary 3.3. *The set of complex diagonalizable matrices is dense*[3] *in $\mathcal{M}_n(\mathbb{C})$.*

Proof. Given a matrix $A \in \mathcal{M}_{p,n}(\mathbb{C})$ and a parameter $\varepsilon > 0$, we know from Theorem 3.1 that there exist $P \in \mathcal{M}_{p,n}(\mathbb{C})$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that:

$$T \equiv P^{-1} A P = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}.$$

Let us then note:

$$A^{(\varepsilon)} \equiv P T^{(\varepsilon)} P^{-1}, \qquad \text{with} \qquad T^{(\varepsilon)} \equiv \begin{pmatrix} \lambda_1 + \varepsilon_1 & & * \\ & \ddots & \\ (0) & & \lambda_n + \varepsilon_n \end{pmatrix},$$

where $\varepsilon_1, \ldots, \varepsilon_n \in (0, \frac{\varepsilon}{\sqrt{n}}]$ are chosen such that the scalars $\lambda_1 + \varepsilon_1, \ldots, \lambda_n + \varepsilon_n$ are all distinct. Lecture 1, Subsection 3.6, Item 2 allows us to set that $T^{(\varepsilon)}$ is diagonalizable, besides:

$$\begin{aligned} \left\| A - A^{(\varepsilon)} \right\| &= \sqrt{\operatorname{Tr}\left(\left(A - A^{(\varepsilon)}\right)^T \left(A - A^{(\varepsilon)}\right) \right)} = \sqrt{\operatorname{Tr}\left(P^{-1} \left(T - T^{(\varepsilon)}\right)^T \left(T - T^{(\varepsilon)}\right) P \right)} \\ &= \left(\operatorname{Tr}\left(\begin{pmatrix} \varepsilon_1 & & * \\ & \ddots & \\ (0) & & \varepsilon_n \end{pmatrix}^T \begin{pmatrix} \varepsilon_1 & & * \\ & \ddots & \\ (0) & & \varepsilon_n \end{pmatrix} \right) \right)^{\frac{1}{2}} = \sqrt{\sum_{i=1}^n \varepsilon_i^2} \leq \varepsilon. \end{aligned}$$

$\square$

Theorem 3.4 (Cayley Hamilton Theorem)**.** $\chi_A(A) = 0$.

Proof. The identity is obvious for diagonalizable matrices. Given a matrix $A \in \mathcal{M}_n(\mathbb{C})$, we know that there exists a sequence of diaonalizabe matrices $(A_n)_{n \in \mathbb{N}} \in \mathcal{M}_n(\mathbb{C})$ such that $\lim A_n = A$. Then the continuity of the determinant (it is only sums and products) provides us the convergence:

$$0 = \lim_{n \to \infty} \chi_{A_n}(A_n) = \chi_A(A),$$

which ends the proof. $\square$

[3]For the metric d defined for any $A, B \in \mathcal{M}_n$ as $d(A, B) = \|A - B\|_F$, since we are in finite dimension, the choice of the norm is not important.

Theorem 3.5. *A square matrix $A \in \mathcal{M}_n(\mathbb{K})$ is diagonalizable iif its minimal annihilating polynomial $P_A \in \mathbb{K}[X]$ splits in $\mathbb{K}[X]$ and has distinct roots*[4].

Remark 3.6. *The matrix A and B below have the same characteristic polynomial (equal to $(X-1)^2$) but their minimal annihilating polynomial are respectively equal to $(X-1)$ and $(X-1)^2$, that is why only A is diagonalizable:*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Proof. The "if" part is a consequence of Bezout Theorem (see Lecture 1, Subsection 3.7 Item 3). Let us note $\lambda_1, \ldots, \lambda_k$, the k distinct roots of the minimal annihilating polynomial of A, P_A and introduce the polynomials $P_1, \ldots, P_k \in \mathbb{K}[X]$ satisfying:

$$\forall i \in [n], P_i = \prod_{\substack{j=0 \\ j \neq i}}^{n} (X - \lambda_j).$$

We know that $P_1, \ldots, P_k$ highest common denominator is 1 (because none of the $\lambda_1, \ldots, \lambda_k$ is root to all $P_1, \ldots, P_k$), therefore, Bezout Theorem (Lecture 1, Subsection 3.7 Item 3) allows us to state the existence of $U_1, \ldots, U_k$ such that:

$$1 = U_1 P_1 + \cdots + U_k P_k,$$

and consequently $I_k = U_1(A)P_1(A) + \cdots + U_k(A)P_k(A)$ (applying A on the right). Now considering $v \in \mathbb{R}^n$, one sees first that:

$$v = U_1(A)P_1(A)v + \cdots + U_k(A)P_k(A)v,$$

which means that $\mathbb{K}^n = \mathrm{Im}(U_1(A)P_1(A)) + \cdots + \mathrm{Im}(U_k(A)P_k(A))$. Besides, for any $i \in [k]$ and $u \in \mathrm{Im}(U_i(A)P_i(A))$, Lecture 1, Subsection 3.7, item 1 allows us to set that there exists $w \in \mathbb{K}$ such that:

$$(A - \lambda_i I_n)u = (A - \lambda_i I_n)U_i(A)P_i(A)w = U_i(A)P_A(A)w = 0,$$

therefore $u \in E_{\lambda_i}$, the eigenspace associated to λ_i. One then has the inclusion sequence:

$$\mathbb{K}^n = \mathrm{Im}(U_1(A)P_1(A)) + \cdots + \mathrm{Im}(U_k(A)P_k(A)) \subset E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_i} \subset \mathbb{K}^n,$$

therefore $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_i} = \mathbb{K}^n$ which implies that A is diagonalizable thanks to Lecture 1, Subsection 3.6, item 4.

Let us now assume that A is diagonalizable to show the "only if" part. There exist k distinct eigenvalues such that $\mathbb{K}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ thanks to Lecture 1, Subsection 3.6, Item 4. Therefore, given $v \in \mathbb{K}^n$, there exist $v_1, \ldots, v_k \in \mathbb{K}^n$ such that $\forall i \in [k]$, $v_i \in E_{\lambda_i}$ and $v = v_1 + \cdots + v_k$. Then, if we note $P = (X - \lambda_1) \cdots (X - \lambda_k)$, one has the identity:

$$P(A)v = P(A)v_1 + \cdots + P(A)v_k = P_1(A)(X - \lambda_1)v_1 + \cdots + P_k(A)(X - \lambda_k)v_k = 0,$$

with the notation $P_1, \ldots, P_k$ given before. Besides, we know from Lecture 1, Subsection 3.7, Item 5, that for all $i \in [k]$, λ_i is a root of P_A, the minimal annihilating polynomial of A. Noting that P divides P_A and annihilates A, one can conclude that $P = P_A$ by definition of P_A. $\square$

[4]That means that there exist k distinct roots $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$ of P_A such that $P_A(X) = (X - \lambda_1) \cdots (X - \lambda_k)$.

Lecture 4

Exponential of matrices and Canonical decomposition

1 Exponential of matrices

Definition 4.1 (Exponential of matrices)**.** *Given a matrix $A \in \mathcal{M}_n(\mathbb{K})$, the exponential of A is noted* $\exp(A)$ *or e^A and defined as:*

$$e^A \equiv \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Example 4.1. 1. ***Diagonal Matrices.*** *Given a diagonal matrix $D = Diag(\lambda_1, \dots, \lambda_n)$ where $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ the exponential of D writes:*

$$e^D A == \sum_{k=0}^{\infty} \frac{D^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & & (0) \\ & \ddots & \\ (0) & & \lambda_n^k \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_1^k & & (0) \\ & \ddots & \\ (0) & & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_n^k \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & & (0) \\ & \ddots & \\ (0) & & e^{\lambda_n} \end{pmatrix}.$$

2. ***Block Diagonal Matrices.*** *The same identity holds for block diagonal matrices, given $A \in \mathcal{M}_n$, $d \in \mathbb{N}$, d integers $n_1, \dots, n_d$ and d matrices $A_1 \in \mathcal{M}_{n_1}, \dots, A_d \in \mathcal{M}_{n_d}$ such that $A = Diag(A_1, \dots, A_n)$:*

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} A_1^k & & (0) \\ & \ddots & \\ (0) & & A_d^k \end{pmatrix} = \begin{pmatrix} e^{A_1} & & (0) \\ & \ddots & \\ (0) & & e^{A_d} \end{pmatrix}.$$

3. ***Diagonalizable Matrices.*** *Given a matrix $A \in \mathcal{M}_n$ such that there exists $P \in \mathcal{M}_n$ invertible satisfying $P^{-1}AP = D$ (with D being the diagonal matrix introduced in item 1), the exponential of A writes:*

$$e^A = \sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} = P \left(\sum_{k=0}^{\infty} \frac{D^k}{k!} \right) P^{-1} = PDiag(e^{\lambda_1}, \dots, e^{\lambda_k})P^{-1}.$$

4. ***Nilpotent matrices.*** *Given a nilpotent matrix of order[1] p, the exponential of N is simply a sum of p elements:*

$$e^N = \sum_{k=0}^{p-1} \frac{N^k}{k!}.$$

[1] It means that $N^{p-1} \neq 0$ and $N^p = 0$.

Proposition 4.2. *Given two matrices $A, B \in \mathcal{M}_n$ such that $AB = BA$, one has the identity:*

$$e^{A+B} = e^A e^B$$

This proposition relies on a the Cauchy product formula given in next Lemma.

Lemma 4.3. *Given two commuting[2] matrices $A, B \in \mathcal{M}_n$, for any $i \in \mathbb{N}$:*

$$(A+B)^i = \sum_{i=0}^{i} \binom{i}{k} A^k B^{i-k},$$

where we recall that $\binom{i}{k} = \frac{i!}{k!(i-k)!}$

proof of Proposition 4.2. Let us simply express regrouping the term with identical total exponent $i = k + l$:

$$\begin{aligned} e^A e^B &= \left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)\left(\sum_{l=0}^{\infty} \frac{B^l}{l!}\right) = \left(I_n + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots\right)\left(I_n + B + \frac{B^2}{2} + \frac{B^3}{3!} + \cdots\right) \\ &= I_n + \underbrace{A+B}_{i=1} + \underbrace{\frac{A^2}{2} + AB + \frac{B^2}{2}}_{i=2} + \underbrace{\frac{A^3}{3!} + \frac{A^2B}{2} + \frac{AB^2}{2} + \frac{B^3}{3!}}_{i=3} + \cdots = \sum_{i=0}^{\infty}\sum_{k=0}^{i} \frac{A^k}{k!}\frac{B^{i-k}}{(i-k)!}. \end{aligned}$$

The Cauchy product formula given in Lemma 4.3 then provides:

$$e^A e^B = \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{k=0}^{i} \binom{i}{k} A^k B^{i-k} = \sum_{i=0}^{\infty} \frac{1}{i!}(A+B)^i = e^{A+B}.$$

□

This proposition will become very useful once one will get the Jordan decomposition introduced in next section. Given $d \in \mathbb{N}$ and $\lambda \in \mathbb{K}$, we denote:

$$J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \lambda \end{pmatrix} \in \mathcal{M}_d. \tag{4.1}$$

Example 4.4. *Note that $J_d(\lambda) = \lambda I_d + J_d(0)$ and for all $k \in [d-1]$, $J_d(0)^k$ is a matrix full of 0 with 1 on the k^{th} upper diagonal and $J_d(0)^d = 0$ ($J_d(0)$ is a nilpotent matrix of degree d). One can express $\forall t \in \mathbb{R}$:*

$$e^{J_d(0)} = \sum_{k=0}^{d-1} \frac{J_d(0)^k}{k!} = \begin{pmatrix} 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(d-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{1}{2!} \\ & (0) & & \ddots & 1 \\ & & & & 1 \end{pmatrix} \tag{4.2}$$

Then, noting that λI_d and $J_d(0)$ commute, one can compute thanks to Proposition 4.2:

$$e^{J_d(\lambda)} = e^{\lambda I_d} e^{J_d(0)} = \begin{pmatrix} e^\lambda & e^\lambda & \frac{e^\lambda}{2!} & \cdots & \frac{e^\lambda}{(d-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{e^\lambda}{2!} \\ & (0) & & \ddots & e^\lambda \\ & & & & e^\lambda \end{pmatrix}. \tag{4.3}$$

[2] It means that $AB = BA$.

Let us end this section with a simple corollary that helps us to compute the inverse of the exponential of a matrix.

Corollary 4.5. *Given* $A \in \mathcal{M}_n(\mathbb{K})$, $(e^A)^{-1} = e^{-A}$.

Proof. It is a simple consequence of Proposition 4.2 since we know that A and $-A$ commute, one can write:

$$I_n = e^0 = e^{A-A} = e^A e^{-A}.$$

□

2 Nilpotent matrices and canonical decomposition

The Jordan decomposition, also called the Canonical decomposition (see Proposition 4.11) is a matrix of the form:

$$\begin{pmatrix} J_{d_1}(\lambda_1) & & (0) \\ & \ddots & \\ (0) & & J_{d_k}(\lambda_k) \end{pmatrix}, \tag{4.4}$$

where some of the $\lambda_1, \ldots, \lambda_k$ are possibly equal and the blocks $J_{d_1}(\lambda_1), \ldots, J_{d_n}(\lambda_n)$ are the Jordan block defined in (4.1).

We will show below that any matrix of $\mathcal{M}_n(\mathbb{K})$ is similar in $\mathcal{M}_n(\mathbb{C})$ to a matrix that satisfies the Jordan decomposition (4.4) (we know from Lecture 3, Remark 3.2, that this is not true in $\mathcal{M}_{p,n}(\mathbb{R})$ because that would mean that real matrices are all triangularizable in $\mathcal{M}_{p,n}(\mathbb{R})$). Recall that $J_{d_i}(\lambda_i)$ is exactly the sum of a diagonal matrix and a nilpotent matrix. A simple proof of the existence nilpotent character of $J_{d_i}(\lambda_i)$ somehow relies on an important result of nilpotent matrices depicted below.

Theorem 4.6. *Given a nilpotent matrix* $N \in \mathcal{M}_{p,n}(\mathbb{K})$ *of degree* d*, considering* $x_0 \in \mathbb{R}^d$ *such that* $h_0 \equiv A^{d-1}x \neq 0$*, note that:*

$$h_0^T A^{d-1} x_0 \neq 0,$$

and of course $(A^T)^{d-1}h_0 \neq 0$*, and* $(A^T)^d = 0$*. Now, if we introduce:*

$$F = \mathbb{K}x_0 + \cdots \mathbb{K}A^{d-1}x_0 \qquad \text{and} \qquad G = \left(\mathbb{K}h_0 + \cdots \mathbb{K}(A^T)^{d-1}h_0\right)^{\perp},$$

we have the two properties:

- $\mathbb{K}^n = F \oplus G$
- *F and G are invariant through A.*

Proof. It is easy to show that $x_0, \ldots, A^{d-1}x_0$ and $h_0, \ldots, (A^T)^{d-1}h_0$ are both linearly independent because A and A^T are both nilpotent of degree d. Let us assume that there exists d scalars $\alpha_0, \ldots, \alpha_{d-1}$ such that: $\alpha_0 x_0 + \cdots + \alpha_{d-1}A^{d-1}x_0 = 0$, then sequentially applying A^{d-1}, A^{d-2} etc.. and A to this equation, we progressively show that $\alpha_0 = 0$, $\alpha_1 = 0$...etc and $\alpha_{d+1} = 0$, which proves that $x_0, \ldots, A^{d-1}x_0$ are linearly independent. The same holds of course for $h_0, \ldots, (A^T)^{d-1}h_0$. As a consequence, we know that $\dim F = k$ and from Lecture 1, Subsection 3.2, Item 2 that $\dim G^T = n - k$.

Now, assuming that there exist $x \in F \subset \{0\}$, $y \in G \subset \{0\}$ and two scalars α, β such that $\alpha x + \beta y = 0$, we know that there exist $k \in [d]$ such that $x = \sum_{i=k}^d \lambda_i A^{i-1}x_0$ and $\lambda_k \neq 0$ then we know that:

$$0 = h_0^T A^{d-k}(\alpha x + \beta y) = \alpha h_0^T \sum_{i=k}^{d} \lambda_i (A^{d-k+i-1}x) = \lambda_i \alpha h_0^T A^{d-1}x$$

(since $y \in (\mathbb{K}h_0^T A^{d-k})^{\perp}$ and $A^{d-k+i-1}x = 0$ for all $i \geq k+1$), then the initial hypothesis $\alpha h_0^T A^{d-1}x \neq 0$ allows us to conclude that $\lambda_i \alpha = 0$ which implies $\alpha = 0$ by hypothesis on λ_i. Of course then, one also has

$\beta = 0$ (since $y \neq 0$) and one has proven that $G^\perp$ and F are in direct sum. Then $\dim(F \oplus G^\perp) = n$ and $F \oplus G^\perp = \mathbb{K}^n$ thanks to Lecture 1, Subsection 3.1, Item 7.

One is left to show that F and $G^\perp$ are A-invariant. Given $x \in G^\perp$, we know that for all $i \in [k]$:

$$(Ax)^T (A^T)^{i-1} h_0 = x^T (A^T)^i h_0 = 0$$

since $x \in (\mathbb{K}(A^T)^i h_0)^\perp$ if $i \leq k-1$ and $(A^T)^k h_0 = 0$. Thus $Ax \in G^\perp$, and we see that G^T is A-invariant. $\square$

Theorem 4.7. *Given a matrix $A \in \mathcal{M}_n(\mathbb{K})$ there exist two A-invariant subspaces $U_1 \subseteq \mathbb{K}^n$ and $U_2 \subseteq \mathbb{K}^n$ with $\mathbb{K}^n = U_1 \oplus U_2$, such that $A|_{U_1}$ is bijective and $A|_{U_2}$ is nilpotent.*

Proof. If $v \in \ker(A)$, then $A^2 v = A(Av) = A(0) = 0$. Thus, $v \in \ker(A^2)$ and therefore $\ker(A) \subseteq \ker(A^2)$. Proceeding inductively, we see that

$$\{0\} \subseteq \ker(A) \subseteq \ker(A^2) \subseteq \ker(A^3) \subseteq \cdots.$$

Since $\mathbb{K}^n$ is finite dimensional, there exists a smallest number $m \in \mathbb{N}_0$ with $\ker(A^m) = \ker(A^{m+j})$ for all $j \in \mathbb{N}$. For this number m let

$$U_1 := \operatorname{im}(A^m), \quad U_2 := \ker(A^m).$$

(If A is bijective, then $m = 0$, $U_1 = \mathbb{K}^n$ and $U_2 = \{0\}$.) We now show that the spaces U_1 and U_2 satisfy the assertion.

First observe that U_1 and U_2 are both A-invariant: If $v \in U_1$, then $v = A^m w$ for some $w \in \mathbb{K}^n$, and therefore $Av = A(A^m w) = A^{m+1} w \in U_1$. If $v \in U_2$, then $A^m v = A^m(A^m w) = A^m(0) = 0$, and therefore $Av \in U_2$.

We have $U_1 + U_2 \subseteq \mathbb{K}^n$. An application of the dimension formula for linear maps to A^m gives $\dim(\mathbb{K}^n) = \dim(U_1) + \dim(U_2)$. If $v \in U_1 \cap U_2$, then $v = A^m w$ for some $w \in \mathbb{K}^n$ (since $v \in U_1$) and hence

$$0 = A^m v = A^m(A^m w) = A^{2m} w.$$

The first equation holds since $v \in U_2$. By the definition of m we have $\ker(A^{2m})$, which implies $A^m w = 0$, and therefore $v = A^m w = 0$. From $U_1 \cap U_2 = \{0\}$ we obtain $\mathbb{K}^n = U_1 \oplus U_2$.

Let us consider $v \in U_1$ such that $Available = 0$. There exists a vector $w \in \mathbb{K}^n$ with $v = A^m w$, which implies $0 = Av = A(A^m w) = A^{m+1} w$. By the definition of m we have $\ker(A^m) = \ker(A^{m+1})$, thus $w \in \ker(A^m)$, and therefore $v = A^m w = 0$. This implies that $\ker(A|_{U_1}) = \{0\}$, i.e., $A|_{U_1}$ is injective and thus also bijective.

Finally, since $U_2 = \ker(A^m)$, for all $v \in U_2$, $A^m v = 0$ which exactly means that A in nilpotent on U_2. $\square$

One can now prove the existence and uniqueness of the Jordan decomposition for any triangularizable matrix

Theorem 4.8. *Any triangularizable matrix $A \in \mathcal{M}_n(\mathbb{K})$ admits a Jordan decomposition. In other words, there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ such that $P^{-1}AP$ follows the decomposition* (4.4) *for some $k \in [n]$, some scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$ possibly equal and some $d_1, \ldots, d_k \in \mathbb{N}_*$.*

Proof. We know from Lecture 3, Theorem 3.1 that A admits at least one eigenvalue $\lambda_1 \in \mathbb{K}$. then setting $B_1 = A - \lambda_1 I_n$, we know from Theorem 4.7 that $\mathbb{K}^n = V_{\lambda_1} \oplus V_{-\lambda_1}$ such that B_1 is stable on V_{λ_1} and $V_{-\lambda_1}$ and $|_{B_1} V_{\lambda_1}$ is nilpotent and $|_{B_1} V_{-\lambda_1}$ is bijective.

Nos we apply Theorem 4.6 to set the existence of two B_1-invariant subspace $U_1^{(\lambda_1)} \subset V_{\lambda_1}$ and $U_{-1}^{(\lambda_1)} \subset V_{\lambda_1}$ such that $V_{\lambda_1} = U_1^{(\lambda_1)} \oplus U_{-1}^{(\lambda_1)}$ and of a vector $x_0 \in V_{\lambda_1}$ such that $U_1^{(\lambda_1)} = \mathbb{K}x_0 + \mathbb{K}B_1 x_0 \oplus \cdots \oplus \mathbb{K}B_1^{\dim U_1^{(\lambda_1)}} x_0$. The decomposition of the matrix $|_{B_1} U_1^{(\lambda_1)}$ in the base $\{x_0, \ldots, B_1^{d_1^{(\lambda_1)}} x_0\}$ (where we noted $d_1^{(\lambda_1)} \equiv \dim(U_1^{(\lambda_1)})$) writes:

$$\begin{pmatrix} 0 & 1 & & (0) \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0, \end{pmatrix} \tag{4.5}$$

and since $A = B_1 + \lambda_1 I_n$ it is also stable on $U_1^{(\lambda_1)}$ and its decomposition on the base $\{x_0^{(\lambda_1)}, \dots, B_1^{d_1^{(\lambda_1)}} x_0^{(\lambda_1)}\}$ is exactly the block $J_{d_1^{(\lambda_1)}}(\lambda_1)$.

The matrix B_1 is also nilpotent on $U_{-1}^{(\lambda_1)})$, one can therefore reproduce the process until one gets a_1 subspaces $U_1^{(\lambda_1)}, \dots, U_{a_1}^{(\lambda_1)}$ and vectors $x_0^{(\lambda_1)} \in U_1^{(\lambda_1)}, \dots, x_{a_1}^{(\lambda_1)}$ such that:

$$V_1 = \mathbb{K}x_1^{(\lambda_1)} \oplus \mathbb{K}B^{d_1^{(\lambda_1)}} x_1^{(\lambda_1)} \oplus \cdots \oplus \mathbb{K}x_{a_1}^{(\lambda_1)} \oplus \mathbb{K}B^{d_{a_1}^{(\lambda_1)}} x_{a_1}^{(\lambda_1)}.$$

The restrict of A in V_1, then writes with this basis decomposition:

$$\begin{pmatrix} J_{d_1^{(\lambda_1)}}(\lambda_1) & & (0) \\ & \ddots & \\ (0) & & J_{d_{a_1}^{(\lambda_1)}}(\lambda_1) \end{pmatrix}$$

Recalling that $\mathbb{K}^n = V_{\lambda_1} \oplus V_{-\lambda_1}$ and that $A - \lambda_1 I_n$ is bijective on $V_{-\lambda_1}$, we know that λ_1 is not an eigenvalue of $|_A V_{-\lambda_1}$, therefore, one consider a new eigenvalue of $|_A V_{-\lambda_1}$ and reproduce the same process done above until one gets a complete Jordan decomposition of A:

$$J = \begin{pmatrix} \begin{array}{ccc} \boxed{J_{d_1^{(\lambda_1)}}(\lambda_1)} & & (0) \\ & \ddots & \\ (0) & & \boxed{J_{d_{a_1}^{(\lambda_1)}}(\lambda_1)} \end{array} & & (0) \\ & \ddots & \\ (0) & & \begin{array}{ccc} \boxed{J_{d_1^{(\lambda_k)}}(\lambda_k)} & & (0) \\ & \ddots & \\ (0) & & \boxed{J_{d_{a_k}^{(\lambda_k)}}(\lambda_k)} \end{array} \end{pmatrix}, \tag{4.6}$$

where $\lambda_1, \dots, \lambda_k$ are all distinct scalars. $\square$

Lemma 4.9. *Given $\lambda \in \mathbb{K}$ and $d \in \mathbb{N}$, the geometric multiplicity of $J_d(\lambda)$ is exactly* 1 *and the algebraic multiplicity of $J_d(\lambda)$ (i.e. the exponent of $(X - \lambda_i)$ as a factor of $\chi_{J_d(\lambda)}$) is equal to d.*

Proof. The algebraic multiplicity is simply deduced from Lecture 1, Subsection 3.4, Item 7 from which we deduce that $\chi_{J_d(\lambda)} = (X - \lambda)^d$. To deduce the geometric multiplicity, let us simply note from the form of $J_d(\lambda) - \lambda I_d$ give in (4.5) that $\ker(J_d(\lambda) - \lambda I_d) = \mathbb{K}e_1$ (and $\mathrm{Im}(J_d(\lambda) - \lambda I_d) = \mathbb{K}e_1 \oplus \cdots \oplus \mathbb{K}e_{k-1}$). $\square$

As a simple consequence, one gets the following lemma (provided without proof):

Lemma 4.10. *Let us consider the matrix J defined in* (4.6)*, if we assume that all the $\lambda_1, \dots, \lambda_k$ are distinct, then for all $i \in [k]$, algebraic multiplicity and the geometric multiplicities associated to λ_i are respectively equal to $\sum_{l=1}^{a_i} d_{a_i}^{(\lambda_i)}$ and a_i.*

The following proposition justifies the uniqueness of the Jordan decomposition and explains why it is often called the "canonical decomposition".

Proposition 4.11. *Given a triangularizable matrix $A \in \mathcal{M}_n(\mathbb{K})$, there exists a unique Jordan decomposition of A up to a permutation of the diagonal blocks.*

Proof. It is a consequence of Lemma 4.10, the fact that the matrices $J_d(\lambda)$ are uniquely defined by λ and d and that similar matrices present equal eigenvalues with equal associated algebraic and geometric multiplicities. $\square$

3 Resolution of linear differential equations

Proposition 4.12. *Given a matrix $A \in \mathcal{M}_n(\mathbb{K})$:*

$$\frac{\partial e^{tA}}{\partial t} = Ae^{tA}.$$

Proof. Let us differentiate:

$$\begin{aligned}\frac{\partial e^{tA}}{\partial t} &= \frac{\partial}{\partial t}\left(I_n + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots\right) \\ &= A + tA^2 + \frac{t^2A^2}{2!} + \cdots = Ae^{tA}.\end{aligned}$$

□

Example 4.13. *Given $d \in \mathbb{N}$ and $t \in \mathbb{R}$, note from (4.2) that:*

$$e^{tJ_d(0)} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{d-1}}{(d-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ (0) & & & & 1 \end{pmatrix}$$

Therefore, one can check that:

$$\frac{\partial e^{tJ_d(0)}}{\partial t} = \begin{pmatrix} 0 & 1 & t & \cdots & \frac{t^{d-2}}{(d-2)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & t \\ & & & \ddots & 1 \\ (0) & & & & 0 \end{pmatrix} = J_d(0)e^{tJ_d(0)}.$$

Theorem 4.14. *The differential equation*[3]*:*

$$\begin{cases} \dot{y} = Ay \\ y(0) = y_0 \in \mathbb{R}^n \end{cases} \tag{4.7}$$

admits as unique solution $y : t \mapsto e^{tA}y_0$.

Example 4.15. *Let us consider a mass $m > 0$ that is fastened to a spring, which possesses a spring constant $\mu > 0$. Denote $x_0 > 0$ as the initial displacement of the mass from its position of equilibrium. We aim to find the position $x(t)$ of a weight at time $t \geq 0$, with the initial position $x(0) = x_0$. Hooke's law governs the extension of the spring, leading to a second-order ordinary differential equation:*

$$\ddot{x} = \frac{d^2x}{dt^2} = -\frac{\mu}{m}x,$$

where $x(0) = x_0$ and $\dot{x}(0) = v_0$, with $v_0 > 0$ representing the initial velocity of the mass. This second-order differential align can be recast as a first-order system by defining v as the velocity, which is the time derivative of x, i.e., $v = \dot{x}$. Consequently, $\dot{v} = \ddot{x}$, and we can represent the system as*

$$\dot{y} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{\mu}{m} & 0 \end{bmatrix}, \quad y = \begin{bmatrix} x \\ v \end{bmatrix}.$$

[3]This formalism means that we look for a differentiable mapping $y : \mathbb{R} \mapsto \mathbb{R}^n$ such that $\forall t \in \mathbb{R}$: $\dot{y}(t) = \frac{\partial y}{\partial t} = y'(t) = Ay(t)$ and $y(0) = y_0$.

Here, the initial condition transforms to $y(0) = y_0 = [x_0, v_0]^T$. According to Theorem 4.14, the solution to this homogeneous initial value problem is uniquely defined by $y = \exp(At)y_0$. We identify the eigenvalues of A as a matrix of $\mathcal{M}_2(\mathbb{C})$ to be two complex numbers $\lambda_1 = i\rho$ and $\lambda_2 = -i\rho$, where $\rho = \sqrt{\frac{\mu}{m}}$. The corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ i\rho \end{bmatrix} \in \mathbb{C}^2, \quad v_2 = \begin{bmatrix} 1 \\ -i\rho \end{bmatrix} \in \mathbb{C}^2.$$

Thus introducing the change of basis matrix $P = (v_1, v_2) \in \mathcal{M}_2(\mathbb{C})$

$$\exp(At)y_0 = S \begin{bmatrix} e^{i\rho t} & 0 \\ 0 & e^{-i\rho t} \end{bmatrix} S^{-1} y_0, \quad \textit{with} \quad S = \begin{bmatrix} 1 & 1 \\ i\rho & -i\rho \end{bmatrix} \in \mathcal{M}_2(\mathbb{C}).$$

Example 4.16. *The Jordan decomposition helps us to solve the differential equation (4.7) for a general $A \in \mathcal{M}_n(\mathbb{C})$ since we know from Theorem 4.8 that there exists $P \in \mathcal{M}_{p,n}(\mathbb{C})$ such that:*

$$P^{-1}AP = \begin{pmatrix} J_{d_1}(\lambda_1) & & (0) \\ & \ddots & \\ (0) & & J_{d_n}(\lambda_n) \end{pmatrix}.$$

One can then compute thanks to Example 4.1, Item 2:

$$e^{tA} = P \begin{pmatrix} e^{tJ_{d_1}(\lambda_1)} & & (0) \\ & \ddots & \\ (0) & & e^{tJ_{d_n}(\lambda_n)} \end{pmatrix} P^{-1} = P \begin{pmatrix} e^{t\lambda_1} e^{tJ_{d_1}(0)} & & (0) \\ & \ddots & \\ (0) & & e^{t\lambda_n} e^{J_{d_n}(0)} \end{pmatrix} P^{-1},$$

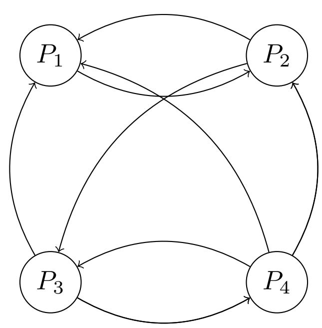
where $e^{tJ_{d_1}(0)}, \dots, e^{tJ_{d_n}(0)}$ have been provided in Example 4.13.

Lecture 5

Largest eigenvalues and Perron Frobenius Theorem

1 Motivation: Page rank algorithm

Let us depict below the general page ranking problem on the internet. Most internet pages are reached through links accessed from other pages. Some pages have few such access links, some have a lot, the ranking system should take into account the difference in popularity of the different pages to provide good advice in search engine like Google.



Let us then denote:

- C_j: the number of outgoing links of page j.
- L_i: the set of pages that refer to page i.
- v_i: importance score of page i.

The score should then satisfy the equation:

$$v_i = \sum_{j \in L_i} \frac{v_j}{C_j}, \quad i = 1, \dots, m$$

This leads to the matrix equation:

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 1 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

The Page Rank problem formalizes in a general setting of n pages:

$$\text{Find } v \in \mathbb{R}^n_+ \text{ s.t.} \quad Av = v \tag{5.1}$$

where the matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$ contains in the column j the values $\frac{1}{C_j}$ at the indexes i such that page j links to page i.

Questions:

1. Does $Av = v$ admit a non-negative solution?
2. Is the solution unique?
3. How to solve $Av = v$?

2 Power method

The power method is a classical technique to find the eigen vector associated to the biggest eigen value of a matrix that has one eigen value with strictly higher modulus. In practice, we compute a sequence $y^{(k)}$ iteratively that should converge to the eigenvector associated to the highest eigenvalue. Given a vector $x \in \mathbb{C}^n$, we note $\nu(x) = x_j$ where $j \in [n]$ is the smallest index such that $|x_j| = \|x\|_\infty$.

Algorithm 1 Compute the eigenvector and eigenvalue iteratively

Consider an initial guess x,
error = 1
while error > tol **do**
  $y = Ax$.
  if $\nu(y) = 0$ **then**
    output $(0, x)$.
  else
    error = $\left\| x - \frac{y}{\nu(y)} \right\|_\infty$
    $x = \frac{y}{\nu(y)}$.
Output $(\nu(y), x)$ as the eigenvalue-eigenvector pair.

If the output is $(0, 0)$, it means that the algorithm was badly initiated, but that never happens when x is chosen uniformly in $\{\mathbb{R}_+^n, \ \|x\|_\infty = 1\}$. Indeed the validity of the method is justified by the following proposition.

Proposition 5.1. *Let us consider a diagonalizable square matrix $A \in \mathcal{M}_n(\mathbb{R})$ that has n (possibly identical) eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \geq 0$, and a basis $\mathcal{B} = (v_1, \ldots, v_n)$ of ν-normalized corresponding eigenvectors (for all $i \in [n]$: $\nu(v_i) = 1$ and $Av_i = \lambda_i v_i$). We consider a vector $y^{(0)} = \sum_{i=1}^n \beta_i v_i \in \mathbb{C}^n$, such that $\beta_1 \neq 0$ and a sequence $(y^{(k)})_{k \in \mathbb{N}}$ satisfying:*

$$\forall k \in \mathbb{N}: \quad y^{(k+1)} = \frac{Ay^{(k)}}{\nu(Ay^{(k)})}, \tag{5.2}$$

where j_k is the smallest index such that $|(Ay^{(k)})_{j_k}| = \|Ay^{(k)}\|_\infty$. The sequence $(y^{(k)})_{k \in \mathbb{N}}$ is well defined ($Ay^{(k)} \neq 0$) tends to v_1 and $e_{j_k}^T Ay^{(k)}$ tends to λ_1.

Be careful, when the dominant eigenvalue of A has an imaginary part different from 0 but $A \in \mathcal{M}_n(\mathbb{R})$, the associated eigenvector also has a non trivial imaginary part, and therefore, one should initialize the power method algorithm with a random complex vector in order to ensure that $\beta_1 \neq 0$.

This proposition relies on two small lemmas on the mapping ν, the first one is quite obvious therefore, we just prove the second one.

Lemma 5.2 (homogeneity of ν)**.** *Given $x \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$: $\nu(\alpha x) = \alpha\nu(x)$.*

Lemma 5.3 (Continuity of ν)**.** *Given a vector $x \in \mathbb{C}^n$ and a sequence of vectors u_k such that $\lim u_k = 0$, we have the convergence $\lim \nu(x + u_k) = \nu(x)$.*

Proof. For k big enough, say for $k > K$, the entries of u_k are so small that the ordering of the entries of x (in modulus) is the same as the ordering of the entries of $x + u_k$. The smallest index j such that $|x_j| = \|x\|_\infty$ is also the smallest index such that $|[x + u_k]_j| = \|x + u_k\|_\infty$, thus for $k \geq K$:

$$\nu(x + u_k) = [x + u_k]_j \underset{k\to\infty}{\longrightarrow} [x]_j = \nu(x).$$

□

Proof of Proposition 5.1. Let us start with the identity:

$$Ay^{(0)} = A\left(\sum_{i=1}^{n} \beta_i v_i\right) = \sum_{i=1}^{n} \beta_i A v_i = \sum_{i=1}^{n} \beta_i \lambda_i v_i,$$

which then implies thanks to Lemma 5.2:

$$y^{(k)} = \frac{Ay^{(k-1)}}{\nu(Ay^{(k-1)})} = \frac{A\frac{Ay^{(k-2)}}{\nu(Ay^{(k-2)})}}{\nu\left(A\frac{Ay^{(k-2)}}{\nu(Ay^{(k-2)})}\right)} = \frac{A^2y^{(k-2)}}{\nu\left(A^2y^{(k-2)}\right)} = \cdots = \frac{A^ky^{(0)}}{\nu(A^ky^{(0)})} = \frac{\sum_{i=1}^{n}\beta_i\lambda_i^k v_i}{\nu(A^ky^{(0)})}.$$

Let us then define:

$$u_k \equiv \sum_{i=1}^{n} \frac{\beta_i}{\beta_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k v_i - v_1,$$

and compute the limit:

$$\|u_k\|_\infty = \left\|\sum_{i=1}^{n} \frac{\beta_i}{\beta_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k v_i - v_1\right\|_\infty = \left\|\sum_{i=2}^{n} \frac{\beta_i}{\beta_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k v_i\right\|_\infty \underset{k\to\infty}{\longrightarrow} 0, \tag{5.3}$$

since $\left(\frac{\lambda_i}{\lambda_1}\right)^k \underset{k\to\infty}{\longrightarrow} 0$ for all $i > 1$. Let us compute the limit:

$$\left\|y^{(k)} - v_1\right\|_\infty = \left\|\frac{A^kx^{(0)}}{\nu(A^kx^{(0)})} - v_1\right\|_\infty = \left\|\frac{\beta_1\lambda_1^k\sum_{i=1}^{n}\frac{\beta_i}{\beta_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k v_i}{\beta_1\lambda_1^k\nu\left(\sum_{i=1}^{n}\frac{\beta_i}{\beta_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k v_i\right)} - v_1\right\|_\infty = \left\|\frac{u_k + v_1}{\nu(u_k + v_i)} - v_1\right\|_\infty \underset{k\to\infty}{\longrightarrow} 0,$$

thanks to (5.3) and Lemma 5.3. One can further add that:

$$\lim_{k\to\infty} \nu(y^{(k)}) = \nu(Av_1) = \lambda_1\nu(v_1) = \lambda_1,$$

again thanks to the continuity of ν given in Lemma 5.3.

□

The Page rank algorithm is used in practice to compute the solution v to the problem (5.1), however, one still needs to prove the existence and uniqueness of such a solution. We will provide in the next two sections some elements of theory concerning the matrix norms and the spectral radius that provide some insights into the highest eigenvalue. In Section 6, we will provide and prove the Perron Frobenius Theorem at the basis of our existence and uniqueness result. In the last section we will explain and justify how is conducted the Page Rank algorithm.

3 Equivalent norms and matrix norms

Recall the definition of a norm on a vectorial space (like $\mathcal{M}_{p,n}$).

Definition 5.1 (Norm).
- *Given a vector space E, a norm $\|\cdot\|$ is a mapping from E to $\mathbb{R}_+$ that should satisfy:*
  1. ***Non-negativity:*** *$\forall u \in E$: $\|u\| \geq 0$.*
  2. ***Positive definiteness:*** *$\forall u \in E$: $\|u\| = 0$ if and only if $u = \mathbf{0}$ (the zero vector).*
  3. ***Scalar Multiplication:*** *$\forall u \in E$, $\alpha \in \mathbb{K}$: $\|\alpha u\| = |\alpha|\|u\|$.*
  4. ***Triangle Inequality:*** *$\forall u, v \in E$: $\|u+v\| \leq \|u\| + \|v\|$.*
- *Two norms $\|\cdot\|$, $\|\cdot\|'$ defined on E are said to be equivalent iif there exists two constants $C, c > 0$ such that:*

$$\forall x \in E : \quad c\|x\| \leq \|x\|' \leq C\|x\|.$$

On finite dimension space (like $\mathcal{M}_{p,n}$), the choice of the norm is not very important thanks to next important result.

Proposition 5.4. *In finite dimension vector space all norms are equivalent.*

This is a famous result in topology[1], therefore, not to go beyond the scope of this course, we just provide partial elements of this proof.

Element of proof. We just show that given a vector space E and a basis $(u_1, \dots, u_n)$ (if $\dim E = n$), all norms are continuous under the norm $\|\cdot\|_1$ defined as:

$$\forall x = \sum_{i=1}^n x_i u_i \in E : \quad \|x\|_1 = \sum_{i=1}^n |x_i|.$$

Indeed, thanks to the triangular inequality, one can bound for the same vector x and for a norm $\|\cdot\|$:

$$\|x\| = \left\| \sum_{i=1}^n x_i u_i \right\| \leq \sum_{i=1}^n |x_i| \, \|u_i\| \leq C\|x\|_1, \qquad \text{with:} \quad C = \max_{i \in [n]} \|u_i\|.$$

The norm $\|\cdot\|_1$ can then be used as a pivot to show that all norms are equivalent. □

This proposition allows us to define the limit in $\mathcal{M}_{p,n}$ without introducing a particular norm.

Definition 5.2 (Limit of matrices). *Given a sequence of matrices $(A_m)_{m \in \mathbb{N}} \in \mathcal{M}_{p,n}$, and a matrix $A \in \mathcal{M}_{p,n}$, we say that $\lim_{m \to \infty} A_n = A$ iif. one of the following properties is satisfied:*

1. ***Given a norm*** *$\|\cdot\|$ on $\mathcal{M}_{p,n}$:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \ s.t. \quad \forall n \geq N : \quad \|A_n - A\| \leq \varepsilon.$$

2. ***For any norm*** *$\|\cdot\|$ on $\mathcal{M}_{p,n}$:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \ s.t. \quad \forall n \geq N : \quad \|A_n - A\| \leq \varepsilon.$$

Definition 5.3 (Matrix norm). *A matrix norm $\|\cdot\|$ on $\mathcal{M}_{p,n}(\mathbb{C})$ is a norm that satisfies for any $A, B \in \mathcal{M}_{p,n}(\mathbb{C})$:*

$$\|AB\| \leq \|A\|\|B\|.$$

[1] See for instance Chapter III, Proposition 7.2. in Choquet, Gustave., and Amiel. Feinstein. Topology. New York: Academic Press, 1966. Print.

Example 5.5. *For any $M \in \mathcal{M}_n$, $p \in [1, \infty]$, we note:*

$$|||M|||_p = \sup \left\{ \frac{\|Mx\|_p}{\|x\|_p}, x \in \mathbb{C} \setminus \{0\} \right\},$$

where we recall that $\forall p \geq 1$, $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ and $\|x\|_\infty = \max_{i\in[n]} |x_i|$. in particular, for any $M \in \mathcal{M}_n$:

$$|||M|||_1 = \max_{j\in[n]} \sum_{i=1}^n |M_{i,j}| \qquad \text{and} \qquad |||M|||_\infty = \max_{i\in[n]} \sum_{j=1}^n |M_{i,j}|.$$

For all $p \in [1, \infty]$, $|||\cdot|||_p$ is a matrix norm since for any $A, B \in \mathcal{M}_n$, one can bound:

$$|||AB|||_p = \sup_{\|x\|_p \leq 1} \|ABx\|_p \leq \sup_{\|x\|_p \leq 1} |||A|||_p \|Bx\|_p \leq \sup_{\|x\|_p \leq 1} |||A|||_p |||B|||_p \|x\|_p \leq |||A|||_p |||B|||_p .$$

Lemma 5.6. *Given a matrix norm $\|\cdot\|$ on $\mathcal{M}_n$, for any invertible matrix $P \in \mathcal{M}_n$, the norm $\|\cdot\|'$ defined for any $M \in \mathcal{M}_n$ as $\|M\|' = \|P^{-1}MP\|$ is also a matrix norm.*

Proof. It is not hard to verify that $\|\cdot\|'$ is a norm, then for any $A, B \in \mathcal{M}_n$:

$$\|AB\|' = \|P^{-1}ABP\| = \|P^{-1}APP^{-1}BP\| \leq \|P^{-1}AP\|\|P^{-1}BP\| = \|A\|'\|B\|'.$$

□

4 Spectral radius

Definition 5.4. *The spectral radius $\rho(A)$ of a matrix $A \in \mathcal{M}_{p,n}(\mathbb{C})$ is defined as:*

$$\rho(A) = \sup \{|\lambda|, \lambda \in Sp(A)\} .$$

When $A \in \mathcal{M}_{p,n}(\mathbb{R}) \subset \mathcal{M}_{p,n}(\mathbb{C})$, one still needs to look at the spectrum in $\mathbb{C}$ to compute the spectral radius $\rho(A)$ because the spectrum in $\mathbb{R}$ could be empty.

Be careful that the spectral radius is not a norm ($\rho(A) = 0 \not\Rightarrow A = 0$, see the lemma below)

Lemma 5.7. *For any nilpotent matrix $A \in \mathcal{M}_m(\mathbb{C})$, $\rho(A) = 0$.*

Proof. Given $x \neq 0$ and $\lambda \in \mathbb{C}$ such that $Ax = \lambda x$, we know that $\forall k \in \mathbb{N}, A^k x = \lambda^k x$. In particular, since $A^n = 0$: $\lambda^n x = A^n x = 0$ which implies $\lambda = 0$. Therefore $\mathrm{Sp}(A) = \{0\}$ and $\rho(A) = 0$. □

Let us give some elementary properties on the spectral radius.

Lemma 5.8. *Given a matrix $A \in \mathcal{M}_n(\mathbb{C})$, an integer $k \in \mathbb{N}$ and a scalar $\alpha \in \mathbb{C}$:*

$$\rho(A)^k = \rho(A^k) \qquad \text{and} \qquad \rho(\alpha A) = |\alpha|\rho(A)$$

It is a simple consequence of the following lemma.

Lemma 5.9. *Given a matrix $A \in \mathcal{M}_n(\mathbb{C})$, an integer $k \in \mathbb{N}$ and a scalar $\alpha \in \mathbb{C}$:*

$$Sp(A^k) = \left\{\lambda^k,\ \lambda \in Sp(A)\right\} \qquad \text{and} \qquad Sp(\alpha A) = \{\alpha\lambda,\ \lambda \in Sp(A)\}$$

Proof. We know from the Schur Theorem that there exist an invertible matrix $P \in \mathcal{M}_n(\mathbb{C})$ and an upper triangular matrix T such that:

$$P^{-1}AP = T = \begin{pmatrix} \lambda_1 & & (*) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are n (possibly identical) eigenvalues of A. Note then that $\mathrm{Sp}(A) = \{\lambda_1, \ldots, \lambda_n\} = \mathrm{Sp}(T)$ and it is immediate to see that $\mathrm{Sp}(\alpha A) = \mathrm{Sp}(\alpha T) = \{\alpha\lambda_1, \ldots, \alpha\lambda_n\}$ and $\mathrm{Sp}(A^k) = \mathrm{Sp}(T^k) = \{\lambda_1^k, \ldots, \lambda_n^k\}$. □

Let us now give a first connection between the spectral radius and matrix norms.

Lemma 5.10. *Given a matrix $A \in \mathcal{M}_{p,n}(\mathbb{C})$, $\rho(A) \leq \|A\|$ for any matrix norm $\|\cdot\|$.*

Proof. There exists $\lambda \in \mathrm{Sp}(A)$ such that $\rho(A) = |\lambda|$, then $Av = \lambda v$ for some $v \in \mathbb{C}^m \setminus \{\mathbf{0}\}$. One can then bound (recall that $e_1 \in \mathbb{R}^n$ is the vector full of zero with a 1 at the first index):

$$\rho(A)\|ve_1^T\| = |\lambda|\|ve_1^T\| = \|\lambda ve_1^T\| = \|Ave_1^T\| \leq \|A\|\|ve_1^T\|,$$

which directly implies our result since $ve_1^T \neq 0$. $\square$

Lemma 5.11. *Given $A \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, there exists a matrix norm $\|\cdot\|'$ such that:*

$$\rho(A) \leq \|A\|' < \rho(A) + \varepsilon$$

Together with Lemma 5.10, this lemma implies:

Corollary 5.12. *$\forall A \in \mathcal{M}_n(\mathbb{C})$: $\rho(A) = \inf_{\|\cdot\|,\ \textit{matrix norm}} \|A\|$.*

Proof of Lemma 5.11. Applying the Schur triangularization to A we know that there exist U unitary and T upper triangular such that:

$$A = UTU^*, \qquad \text{with:} \quad T = \begin{bmatrix} \lambda_1 & T_{1,2} & \dots & T_{1,n} \\ 0 & \lambda_2 & T_{2,3} & \ddots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{bmatrix}$$

For all $m \in \mathbb{N}$, define a norm $\|\cdot\|_m$ followingly:

$$\forall B \in \mathcal{M}_n(\mathbb{C}): \quad \|B\|_m \equiv \left|\!\left|\!\left| D_m^{-1}U^*BUD_m \right|\!\right|\!\right|_1 \qquad \text{where}: \quad D_m = \mathrm{Diag}(\frac{1}{m}, \dots, \frac{1}{m^n})$$

where $|\!|\!|\cdot|\!|\!|_1$ is the matrix norm defined in Example 5.5 (for all $M \in \mathcal{M}_m(\mathbb{C})$ $|\!|\!|M|\!|\!|_1 = \sup_{j\in[n]} \sum_{i=1}^n |M_{i,j}|$). We know Lemma 5.6 that for any $m \in \mathbb{N}$, $\|\cdot\|_m$ is a norm and:

$$\begin{aligned} \|A\|_m = \left|\!\left|\!\left| D_m^{-1}TD_m \right|\!\right|\!\right|_1 &= \left|\!\left|\!\left| \begin{bmatrix} \lambda_1 & T_{1,2}m^{-1} & \dots & T_{1,n}m^{-n+1} \\ 0 & \lambda_2 & \dots & T_{2,m}m^{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{bmatrix} \right|\!\right|\!\right|_1 \\ &= \max\left(|\lambda_1|, |\lambda_2| + \frac{|T_{1,2}|}{m}, \dots, |\lambda_n| + \frac{|T_{1,n}|}{m^{1-n}} + \dots + \frac{|T_{n-1,n}|}{m} \right) \\ &\leq \rho(A) + \frac{|\!|\!|T|\!|\!|_1}{m} \underset{m\to\infty}{\longrightarrow} \rho(A). \end{aligned}$$

Therefore, there exist m big enough such that:

$$\rho(A) \leq \|A\|_m \leq \rho(A) + \varepsilon.$$

$\square$

Lemma 5.13. *Given a matrix $A \in \mathcal{M}_n$:*

$$\lim_{k\to\infty} A^k = 0 \iff \rho(A) < 1$$

(the limit of sequence of matrices has been defined in Definition 5.2)

Proof. Consider $x \in \mathbb{C}^n$ such that $Ax = \lambda x$ with $\rho(A) = |\lambda|$. One has:

$$\rho(A)^k \|x\| = |\lambda|^k \|x\| = \|\lambda^k x\| = \|A^k x\| \underset{k\to\infty}{\longrightarrow} 0,$$

thus $\rho(A) < 1$.

Let us now assume $\rho(A) < 1$ and set $\varepsilon = \frac{1-\rho(A)}{2}$. By Lemma 5.11, there exists a matrix norm such that $\|A\|' \leq \rho(A) + \varepsilon < 1$, then:

$$\|A^k\|' \leq (\|A\|')^k \leq (\rho(A) + \varepsilon)^k \underset{k\to\infty}{\longrightarrow} 0,$$

since all the norms are equivalent in $\mathcal{M}_n$, that means that $A^k \underset{k\to\infty}{\longrightarrow} 0$. □

Theorem 5.14. *For any matrix norm $\|\cdot\|$, we have*

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}$$

Proof. On the first hand Lemmas 5.8 and 5.10 allow us to bound:

$$\rho(A) = \rho(A^k)^{\frac{1}{k}} \leq \|A^k\|^{\frac{1}{k}},$$

and in particular $\rho(A) \leq \liminf_{k\to\infty} \|A^k\|^{\frac{1}{k}}$.

Given $\varepsilon > 0$, let us introduce $\tilde{A} = \frac{A}{\rho(A)+\varepsilon}$, then $\rho(\tilde{A}) < 1$ and by Lemma 5.13, there exists N such that for all $k > N$:

$$\|A^k\| \leq \|\tilde{A}^k\| \left(\rho(A) + \varepsilon\right)^k \leq (\rho(A) + \varepsilon)^k.$$

This inequality being true for any $\varepsilon > 0$, one finally obtains the inequality:

$$\forall l \in \mathbb{N}: \quad \limsup_{k\to\infty} \|A^k\|^{\frac{1}{k}} \leq \rho(A) \leq \liminf_{k\to\infty} \|A^k\|^{\frac{1}{k}}.$$

In other words, $\lim_{k\to\infty} \|A^k\|^{\frac{1}{k}} = \rho(A)$. □

5 Positive matrices

We say that a matrix $A \in \mathcal{M}_n(R)$ is *entry-positive* if all its entries are positive (i.e. if $\forall i, j \in [n]$, $A_{i,j} \geq 0$). Be careful we will define later in this course the notion of *symmetric positive* matrices (resp. hermitian positive matrices) which designates symmetric matrices (resp. hermitian matrices[2]) $A \in \mathcal{M}_n(\mathbb{R})$ (resp. $A \in \mathcal{M}_n(\mathbb{C})$) such that $\forall x \in \mathbb{R}^n$ (resp. $\forall x \in \mathbb{C}^n$), $x^T A x \geq 0$ (resp. $x^* A x \geq 0$). Given two matrices $A, B \in \mathcal{M}_n(\mathbb{R})$ we further note $A \geq B$ if $\forall i, j \in [n]$, $A_{i,j} \geq B_{i,j}$ and $|A| = (|A_{i,j}|)_{i,j\in\mathbb{N}} \in \mathcal{M}_n(\mathbb{R}_+)$.

Lemma 5.15. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $A \leq B$ entry-wise and $B \geq 0$. Then $\rho(A) \leq \rho(|A|) \leq \rho(B)$.*

Proof. Since $A \leq |A| \leq B$, we have $A^k \leq |A|^k \leq B^k$. This implies $\|A^k\|_F \leq \||A|^k\|_F \leq \|B^k\|_F$. By Theorem 5.14, $\rho(A) \leq \rho(|A|) \leq \rho(B)$. □

Given a matrix $A \in \mathcal{M}_n$, we call a *submatrix* of A any matrix $\hat{A} \in \mathcal{M}_n$ such that there exist two index sets $I, J \subset [n]$ satisfying:

$$\hat{A}_{i,j} = \begin{cases} A_{i,j} & \text{if } i \in I \text{ and } j \in J \\ 0 & \text{otherwise.} \end{cases}$$

Then the following corollary is a simple consequence of Lemma 5.15 applied with $B = \hat{A} \leq A$.

[2]Recall that it means that $A^* = A$.

Corollary 5.16. *Given an entry-positive matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$ and $\hat{A} \in \mathcal{M}_n(\mathbb{R}_+)$, a submatrix of A, we have $\rho(\hat{A}) \leq \rho(A)$.*

Lemma 5.17. *Given an entry-positive matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$, if the row sums of A are equal, then*[3] $\rho(A) = \|A\|_\infty$*. If the column sums of A are equal, then $\rho(A) = \|A\|_1$.*

Proof. If all the rows of A sum to $\|A\|_\infty$, then $A\mathbb{1} = \|A\|_\infty \mathbb{1}$ and $\|A\|_\infty \leq \rho(A)$ but we know that $\|\cdot\|_\infty$ is a matrix norm, therefore $\|A\|_\infty \geq \rho(A)$ thanks to Lemma 5.10 and finally $\rho(A) = \|A\|_\infty$. If the columns are equal, we obtain the same result considering A^T since A^T having same eigenvalues as A, $\rho(A) = \rho(A^T) = \|A\|_1$ (note that then the eigenvector associated to $\rho(A)$ is not $\mathbb{1}$). ☐

Lemma 5.18. *Given an entry-positive matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$:*

$$\min_i \sum_j A_{ij} \leq \rho(A) \leq \|A\|_\infty = \max_i \sum_j A_{ij}$$

Proof. Denote $\alpha = \min_i \sum_j A_{ij}$. If $\alpha = 0$, then it is true. Suppose $\alpha > 0$ and construct a matrix $B \in \mathcal{M}_n$ such that:

$$B_{i,\cdot} = \frac{\alpha}{\sum_{j=1}^n A_{i,j}} A_{i,\cdot}$$

where $A_{i,\cdot}$ (resp. $B_{i,\cdot}$) is the i-th row of A (resp. of B). By Lemma 5.17, $\alpha = \rho(B) \leq \rho(A)$. ☐

Theorem 5.19. *Given an entry-positive matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$, for any $x \in \mathbb{R}_+^n$ (with positive entries), we have:*

$$\min_{i \in [n]} \frac{1}{x_i} \sum_{j=1}^n A_{i,j} x_j \leq \rho(A) \leq \max_{i \in [n]} \frac{1}{x_i} \sum_{j=1}^n A_{i,j} x_j$$

Proof. Denoting $S = \mathrm{diag}(x) = \mathrm{diag}(x_1, x_2, \ldots, x_m)$ one can conclude with Lemma 5.18 applied to $S^{-1}AS$ since $\rho(S^{-1}AS) = \rho(A),$. ☐

Corollary 5.20. *Given an entry-positive matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$, $x \in \mathbb{R}_+^n$ and $\alpha, \beta \geq 0$ one has the implications:*

$$\begin{cases} \alpha x \leq Ax \leq \beta x & \Longrightarrow \quad \alpha \leq \rho(A) \leq \beta, \\ \alpha x < Ax < \beta x & \Longrightarrow \quad \alpha < \rho(A) < \beta. \end{cases}$$

Proof. Given $i \in [n]$, one has:

$$\forall i \in [n] : \frac{1}{x_i} \sum_{j=1}^n A_{i,j} x_j = \frac{(Ax)_i}{x_i} \leq \beta,$$

in particular, Theorem 5.19 allows us to set that $\rho(A) \leq \beta$ and one can show similarly that $\alpha \leq \rho(A)$. The implication between strict inequalities, is shown the same way. ☐

Corollary 5.21. *Given an entry-positive matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$, the eigenvectors with positive entries are associated to the eigenvalue $\rho(A)$.*

Proof. Considering $x \in \mathbb{R}_+^n$ such that $Ax = \lambda x$, one knows that $\lambda \in \mathbb{R}_+$, Corollary 5.20 (and inequality $\lambda x \leq Ax \leq \lambda x$) then allow us to conclude that $\lambda \leq \rho \leq \lambda$, in other words, $\rho(A) = \lambda$. ☐

[3] Recall that $\|\cdot\|_\infty$ in the case of a entry-positive matrix is the max of the rows: $\|A\|_\infty = \max_{i \in [n]} \sum_j = 1^n A_{i,j}$

6 Perron Frobenus Theorem

The next three theorems are all results of the so-called Perron-Frobenius Theorem.

Theorem 5.22. *Given a matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$ such that $A > 0$ and a vector $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $Ax = \lambda x$ for some $|\lambda| = \rho(A)$, we have the identity:*

$$A|x| = \rho(A)|x| \qquad \text{and} \qquad |x| > 0$$

Note that this theorem implies:

1. $\rho(A)$ is an eigenvalue of A,
2. $\rho(A) > 0$,
3. there is a positive eigenvector associated with $\rho(A)$.

Proof of Theorem 5.22. Given $Ax = \lambda x$ with $|\lambda| \leq \rho(A)$, we have $A|x| \geq |Ax| = |\lambda x| = \rho(A)|x|$. If $A|x| > \rho(A)|x|$ then Corollary 5.20 would imply $\rho(A) < \rho(A)$, which is impossible since $\rho(A) \geq \max_i A_{ii} > 0$ by Corollary 5.16. Therefore, $A|x| = \rho(A)|x|$ and $\forall i \in [n]$: $|x|_i = \frac{1}{\rho(A)} \sum_{j=1}^n A_{i,j}|x|_j > 0$ since $|x| \neq 0$ and $A > 0$. ☐

Lemma 5.23. *Given a matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$ such that $A > 0$, $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $Ax = \lambda x$ with $|\lambda| = \rho(A)$, then for some $\theta \in \mathbb{R}$, $e^{i\theta}x = |x| > 0$.*

Proof. The triangular inequality of the modulus provides:

$$\forall i \in [n]: \quad |Ax_i| = \left| \sum_{j=1}^m A_{ij} x_j \right| \leq \sum_{j=1}^m A_{ij}|x_j| = (A|x|)_i \tag{5.4}$$

Besides:

$$|Ax| = |\lambda||x| = \rho(A)|x| = A|x|,$$

where the last equality is a consequence of Theorem 5.22. Therefore the triangular inequality in (5.4) is reached, which implies that there exists $\theta \in \mathbb{R}$ such that for all $i \in [n]$, $x_i = e^{\theta \mathbf{i}}|x_i|$. ☐

Theorem 5.24. *Given a matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$ such that $A > 0$, $\{\lambda \in Sp(A), |\lambda| = \rho(A)\} = \{\rho(A)\}$.*

Proof. Suppose we have a λ such that $|\lambda| = \rho(A)$ and $Ax = \lambda x$ for some $x \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. By Lemma 5.23, there exists $w = e^{i\theta}x \geq 0$. Then:

$$Aw = e^{i\theta}Ax = \lambda(e^{i\theta}x) = \lambda w,$$

and by Corollary 5.21, $\lambda = \rho(A)$. ☐

Theorem 5.25. *Given a matrix $A \in \mathcal{M}_n(\mathbb{R}_+)$ such that $A > 0$, $\dim(Ker(A - \rho(A)I_n) = 1$.*

Proof. Suppose we have $Aw = \rho(A)w$, $Az = \rho(A)z$ and $w \neq 0$, $z \neq 0$. By Lemma 5.23, there exist $\theta, \sigma \in \mathbb{R}$ such that, if one notes $\omega = e^{\theta \mathbf{i}}w$ and $\zeta = e^{\sigma \mathbf{i}}z$:

$$A\omega = \rho(A)\omega \qquad A\zeta = \rho(A)\zeta \qquad \text{and} \qquad \omega, \zeta > 0.$$

Let us then denote $\alpha = \min_{i \in [n]} \frac{\omega_i}{\zeta_i}$ and set $x = \omega - \alpha\zeta$. Then $Ax = A\omega - \alpha A\zeta = \rho(A)x$, which implies $x = \mathbf{0}$ because if $x \neq \mathbf{0}$, then Theorem 5.22 would imply $x > 0$ which is impossible since $x_i = 0$. Therefore, $\omega = \alpha\zeta$, which means z and p are aligned (in $\mathbb{C}^n$), and $\dim(\mathrm{Ker}(A - \rho(A)I_n) = 1$. ☐

7 Application to Page Rank algorithm

In the context of the PageRank algorithm, the original matrix $A \in \mathcal{M}_n(\mathbb{R}^+)$ is column-stochastic (i.e. all the columns sum to 1) but may not be strictly positive due to zero entries. To address this and create a problem with a unique solution, consider the following approach:

Define $S = \frac{1}{n}\mathbb{1}\mathbb{1}^T \in \mathcal{M}_n(\mathbb{R}^+)$, it is clear that S is positive and column-stochastic. Given $\alpha \in (0,1)$, we introduce the matrix:

$$\hat{A}(\alpha) = (1-\alpha)A + \alpha S.$$

We know from Lemma 5.17 that $\rho(A) = 1$ and Theorem 5.25 implies the existence of a unique positive eigenvector $\hat{u}$ such that $\|u\|_1 = 1$ and corresponding to the eigenvalue $\rho(A) = 1$. The relationship is described as:

$$\hat{u} = \hat{A}(\alpha)\hat{u} = (1-\alpha)A\hat{u} + \alpha S\hat{u} = (1-\alpha)A\hat{u} + \frac{\alpha}{n}\mathbb{1}^T.$$

Considering a vast set of documents, such as the internet, the fraction $\frac{\alpha}{n}$ becomes negligible. Hence, the vector $\hat{u}$ approximates an eigenvector u of the original matrix A with $Au \approx u$. Solving for $\hat{u}$ in $\hat{A}(\alpha)\hat{u} = \hat{u}$ serves as a practical solution to finding u, and this problem can be solved with the power method presented in Section 2.

Lecture 6

Symmetric / Hermitian matrices and their eigenvalues

1 General properties and definitions

Definition 6.1. *A matrix $A \in \mathcal{M}_n(\mathbb{R})$ is said to be symmetric iif. $A^T = A$. A matrix $H \in \mathcal{M}_n(\mathbb{C})$ is said to be Hermitian iif. $H^* = H$. We will note $\mathcal{H}_n$ the set of Hermitian matrices of $\mathcal{M}_n(\mathbb{C})$.*

Note that if $A \in \mathcal{M}_n(\mathbb{R})$ is hermitian, then it is symmetric, therefore, some of the coming properties expressed only for Hermitian matrices are also valid for such symmetric matrices A.

Proposition 6.1. *The eigenvalues of a Hermitian matrix are real values.*

Proof. Considering $A \in \mathcal{H}_n$, if there exist $v \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ such that $Av = \lambda v$, then:

$$\lambda v^* v = v^* A v = v^* A^* v = \lambda^* v^* v$$

which implies $\lambda = \lambda^*$ since $v \neq 0$, hence λ is real valued. □

Proposition 6.2. *Given λ_1, λ_2, 2 eigenvalues of a hermitian matrix $A \in \mathcal{H}_n$, if $\lambda_1 \neq \lambda_2$ the eigenspaces $E_{\lambda_1} = \ker(A - \lambda_1 I_n)$, $E_{\lambda_2} = \ker(A - \lambda_2 I_n)$ are orthogonal.*

Proof. Given $v_1 \in E_{\lambda_1}$ and $v_2 \in E_{\lambda_2}$, one has the identities:

$$v_1^* A v_2 = \lambda_2 v_1^* v_2 \qquad \text{and} \qquad v_1^* A v_2 = (A^* v_1)^* v_2 = (A v_1)^* v_2 = \lambda_1^* v_1^* v_2 = \lambda_1 v_1^* v_2$$

One then obtains the equality $(\lambda_1 - \lambda_2) v_1^* v_2 = 0$ which directly implies $v_1^* v_2 = 0$ since $\lambda_1 \neq \lambda_2$. □

Theorem 6.3. *Any Hermitian matrix is diagonalizable with unitary matrices.*

Proof. By Schur Triangulation, there exist $U \in \mathcal{M}_n(\mathbb{C})$ unitary ($U^* = U^{-1}$) and $T \in \mathcal{M}_n(\mathbb{C})$ such that $A = U^* T U$. One then has the identity:

$$T^* = U A^* U^* = U A U^* = T,$$

which implies T diagonal since T is upper triangular and T^* is lower triangular. □

Definition 6.2. *A matrix $A \in \mathbb{H}^n$ is positive semi-definite (PSD) iif*

$$\forall x \in \mathbb{C}^n : \quad x^* A x \geq 0.$$

It is positive definite (PD) iif:

$$\forall x \in \mathbb{C}^n \setminus \{0\} : \quad x^* A x > 0.$$

We denote $A \succeq 0$ or $A \succ 0$ for $A \in \mathcal{H}_n$ respectively PSD and PD.

Example 6.4.

- ***(Covariance Matrix)***: *Given a random vector $Y \in \mathbb{C}^n$ such that $\mathbb{E}[Y] = 0$ we denote R_Y its covariance, it is defined as:*
$$R_Y \equiv \mathbb{E}[YY^*]$$
The covariance is PSD as for any $x \in \mathbb{C}^n$: $x^* R_Y x = x^* \mathbb{E}[YY^*]x = \mathbb{E}[x^* YY^* x] = \mathbb{E}[|Y^* x|^2] \geq 0$.

- ***(Hessian Matrix)*** *Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we denote the gradient vector:*
$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix} \in \mathbb{R}^n,$$
and the Hessian matrix:
$$\nabla^2 f(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right]_{i,j\in[n]} \in \mathcal{M}_n(\mathbb{R})$$
A well known theorem sets that $\nabla^2 f(x)$ is symmetric if f is twice differentiable and continuous (we say that f is of class $\mathcal{C}^2$). More over $\nabla^2 f(x)$ is PSD iif. f is a convex function. For instance, considering $f : x \mapsto x^* Ax + 2b^T x + c$ (f is a quadratic function), one can compute:
$$\nabla f(x) = 2Ax + 2b \qquad \text{and} \qquad \nabla^2 f(x) = 2A.$$
If A is PSD, $f(x)$ is convex.

- ***(Ellipsoid)*** *Consider the ellipsoid, which is the set of points $x = (x_1, x_2) \in \mathbb{R}^2$ such that:*
$$\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 \leq 1,$$
where a_1 and a_2 denote the shape of the ellipsoid. This can be written in matrix form as
$$x^T \begin{pmatrix} \frac{1}{a_1^2} & 0 \\ 0 & \frac{1}{a_2^2} \end{pmatrix} x \leq 1.$$
In dimension n, given any $P \in \mathcal{M}_n(\mathbb{R}^n)$ PD, the set $\mathcal{E} = \{x \in \mathbb{R}^n \mid x^T Px \leq 1\}$ defines an ellipsoid. With the eigenvalue decomposition of P: $P = V\Lambda V^T$ with V orthogonal and Λ diagonal, we have
$$x^T Px = x^T V\Lambda V^T x = z^T \Lambda z,$$
where $z = V^T x$ is a rotation of x.

Proposition 6.5. *If A is PSD, then any principal submatrix[1] is PSD.*

As a consequence any square diagonal block of a PSD matrix is PSD (in particular the diagonal entries are positive).

Proof. Given $I = \{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, n\}$, for any $x \in \mathbb{C}^k$, we introduce $\tilde{x} \in \mathbb{C}^n$ such that $\forall l \in [k]$, $\tilde{x}_{i_l} = x_l$, and $\forall i \notin \{i_1, \dots, i_k\}$, $x_i = 0$. One can then express:
$$x^* A_I x = \tilde{x}^* A\tilde{x} \geq 0.$$
□

[1] A principal supmatrix of $A \in \mathcal{M}_n(\mathbb{C})$, is any matrix written $A_I = (A_{i,j})_{i,j\in I} \in \mathcal{M}_k(\mathcal{C})$, where $I = \{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, n\}$.

2 Root of Hermitian matrices

Problem 6.6. *Given $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathcal{S}^n$, PD, we denote $X \sim \mathcal{N}(\mu, \Sigma)$ if the random vector $X \in \mathbb{R}^m$ follows a multivariate Gaussian distribution with probability density function given by:*

$$\phi(x) = \frac{1}{(2\pi)^{\frac{m}{2}} \det(\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$$

note then that $\mu = \mathbb{E}[x]$ and $\Sigma = \mathbb{E}[(x-\mu)(x-\mu)^T]$.

One can show easily that $X - \mu \sim \mathcal{N}(0, \Sigma)$ and has a density:

$$\phi(x) = \frac{1}{(2\pi)^{\frac{m}{2}} \det(\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}x^T \Sigma^{-1} x\right).$$

Question: *Does there exist $S \in \mathcal{M}_n(\mathbb{C})$ such that $S(X-\mu) \sim \mathcal{N}(0, I_n)$?*

Thanks to Theorem 6.8 settled below, we can consider $S \equiv \Sigma^{-\frac{1}{2}}$, one can then note that for any bounded mapping $f : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned}
\int_{\mathbb{R}^n} f(S(x-\mu))\phi(x)dx &= \int_{\mathbb{R}^n} \frac{f(S(x-\mu))}{(2\pi)^{\frac{m}{2}} \det(\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right) dx \\
&= \int_{\mathbb{R}^n} \frac{f(y)}{(2\pi)^{\frac{m}{2}}} \exp\left(-\frac{1}{2}y^T y\right) dy
\end{aligned}$$

thanks to the change of variable:

$$\begin{cases} y = Sx + \mu \\ dy = \det(S)dx = \det(\Sigma)^{-\frac{1}{2}} dx. \end{cases}$$

We see that the density of $S(X-\mu)$ is the same as the density of a Gaussian random vector with zero mean and identity covariance matrix.

Theorem 6.8 setting the existence and uniqueness of S in the previous problem requires the following property in order to set the uniqueness.

Proposition 6.7. *Given two Hermitian matrices $A, B \in \mathcal{H}_n$, if $AB = BA$, then there exist a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that U^*AU and U^*BU are both diagonal. One says that A and B are co-diagonalizable.*

Proof. Let us consider $\lambda_1, \ldots, \lambda_k$, k distinct eigenvalues of B such that $\mathrm{Sp}(B) = \{\lambda_1, \ldots, \lambda_k\}$, we further denote for all $i \in [k]$, $E_i \equiv \ker(A - \lambda_i I_n)$, the eigenspace associated to λ_i, and $v_1^{(i)}, \ldots, v_{d_i}^{(i)}$ an orthogonal basis of the eigenspace E_i, where of course $d_i = \dim(E_i)$. Noting $V = (v_1^{(1)}, \ldots, v_{d_1}^{(1)}, \ldots, v_1^{(k)}, \ldots, v_{d_k}^{(k)}) \in \mathcal{M}_n(\mathbb{C})$, we know that V is unitary and diagonalizes B:

$$V^*BV = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_k & & \\ & & & & & \ddots & \\ & & & & & & \lambda_k \end{pmatrix}$$

Given $i \in [k]$ and any $v \in E_i$, the commutation hypothesis allows us to set $BAv = ABv = \lambda_i Av$. We see that $Av \in E_i$, thus, there exist k matrices $A_1, \ldots, A_k$ such that:

$$V^*AV = \begin{pmatrix} A_1 & & (0) \\ & \ddots & \\ (0) & & A_k \end{pmatrix},$$

and since $(V^*AV)^* = V^*A^*V$, we know $A_1, \ldots, A_k$ are all hermitian. Therefore for all $i \in [k]$, there exists a unitary matrix $U_i \in \mathcal{M}_n$ such that $U_i^* A_i U_i = D_i$ where D_i is diagonal. Let us then introduce the unitary matrix:

$$U \equiv \begin{pmatrix} U_1 & & (0) \\ & \ddots & \\ (0) & & U_k \end{pmatrix}, \qquad \text{note that:} \quad U^* \equiv \begin{pmatrix} U_1^* & & (0) \\ & \ddots & \\ (0) & & U_k^* \end{pmatrix}$$

Then we have naturally the identities:

$$\begin{cases} U^*V^*AVU = \begin{pmatrix} D_1 & & (0) \\ & \ddots & \\ (0) & & D_k \end{pmatrix} \\ U^*V^*BVU = \begin{pmatrix} U_1^*(\lambda_1 I_{d_1})U_1 & & (0) \\ & \ddots & \\ (0) & & U_k^*(\lambda_k I_{d_k})U_k \end{pmatrix} = \begin{pmatrix} \lambda_1 I_{d_1} & & (0) \\ & \ddots & \\ (0) & & \lambda_k I_{d_k} \end{pmatrix}, \end{cases}$$

this is exactly what we wanted to prove since VU is unitary ($(VU)^*VU = U^*V^*VU = I_n$). ☐

Theorem 6.8. *Given an hermitian matrix $A \in \mathcal{H}_n$ PSD, and $k \in \mathbb{N}$, there exists a unique matrix $B \in \mathcal{H}_n$ such that $B^k = A$. We denote $B = A^{\frac{1}{k}}$.*

Proof. The existence is straight forward. We know that there exits $U \in \mathcal{M}_n(\mathbb{C})$ such that $U^*AU = \Lambda$ with $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ and $\lambda_1, \ldots, \lambda_n \geq 0$. One can then introduce $B = U\mathrm{Diag}(\lambda_1^{\frac{1}{k}}, \ldots, \lambda_n^{\frac{1}{k}})U^* = U\Lambda^{\frac{1}{k}}U^*$. It satisfies $B^k = (U\Lambda^{\frac{1}{k}}U^*)^k = U\Lambda U^*$.

To show the uniqueness, let us assume there exists a second matrix $C \in \mathcal{M}_n$ such that $C^k = A$. Assuming that $\mathrm{Sp}(A) = \{\mu_1, \ldots, \mu_l\} (= \{\lambda_1, \ldots, \lambda_n\})$ with $\mu_1, \ldots, \mu_l$ all different from one another, let us introduce the polynomial:

$$P(X) = \sum_{i=1}^{l} \mu_i^{\frac{1}{k}} \frac{(X-\mu_1)\cdots(X-\mu_{i-1})(X-\mu_{i+1})\cdots(X-\mu_l)}{(\mu_i-\mu_1)\cdots(\mu_i-\mu_{i-1})(\mu_i-\mu_{i+1})\cdots(\mu_i-\mu_l)}.$$

Note that for all $i \in [l]$ $P(\mu_l) = \mu_l^{\frac{1}{k}}$. With this choice:

$$P(A) = UP(\Lambda)U^* = U \begin{pmatrix} P(\lambda_1) & & (0) \\ & \ddots & \\ (0) & & P(\lambda_n) \end{pmatrix} U^* = U \begin{pmatrix} \lambda_1^{\frac{1}{k}} & & (0) \\ & \ddots & \\ (0) & & \lambda_n^{\frac{1}{k}} \end{pmatrix} U^* = B$$

Now, since $C^k = A$ we see that $B = P(A) = P(C^k)$ commutes with C as a polynomial of C. Therefore, by Proposition 6.7, there exists $V \in \mathcal{M}_n(\mathbb{C})$, unitary, such that:

$$V^*BV = \Gamma_B \qquad \text{and} \qquad V^*CV = \Gamma_C$$

with $\Gamma_B, \Gamma_C \in \mathcal{M}_n(\mathbb{R}_+)$, both diagonal. Now:

$$\Gamma_B^k = V^*B^kV = V^*AV = V^*C^kV = \Gamma_C^k,$$

thus, by uniqueness of the k^{th} root in $\mathbb{R}_+$, we deduce that $\Gamma_C = \Gamma_B$ and, as a consequence $B = C$, the k^{th} root of A is unique. ☐

3 Order relation and minmax formulas for eigenvalues

Let us introduce in the set of hermitian matrices the following order relation:

$$A \succeq B \qquad \iff \qquad A - B \succeq 0$$

We present below a set of important properties to deal with these inequalities.

Lemma 6.9. *We consider $A, B, C \in \mathcal{H}_n$, $\alpha \in \mathbb{C}$ and we note $\lambda_1(A), \ldots, \lambda_n(A)$ the eigenvalues of A:*

1. *If $A \succeq 0, \alpha \geq 0$ then $\alpha A \succeq 0$.*
2. *If $A \succeq 0, B \succeq 0$ then $A + B \succeq 0$.*
3. *If $A \succeq B, B \succeq C$ then $A \succeq C$.*
4. *Given any invertible matrix[2] $P \in \mathcal{M}_{p,n}$: $P^*AP \succeq B \implies A \succeq P^{-*}BP^{-1}$ and $P^*AP \succ B \implies A \succ P^{-*}BP^{-1}$.*
5. *$A \succeq B \succ 0 \iff 0 \prec A^{-1} \preceq B^{-1}$.*
6. *If $A \succeq I_n$, then $\lambda_i(A) \geq 1 \quad \forall i = 1, \ldots, m$.*

Proof. *4.* If $P^*AP \succeq B$, for all $y \in \mathbb{C}^n$, $y^*(P^*AP - B)y \geq 0$ and in particular, for any $x \in \mathbb{C}^n$, replacing y with $P^{-1}x$, one obtains:

$$0 \leq (P^{-1}x)^*(P^*AP - B)P^{-1}x = x * (A - P^{-*}BP^{-1})x$$

The second inequality is just proven with strict inequalities replacing large inequalities.

5. Let us first treat the case $B = I_n$. Let us decompose thanks to Theorem 6.8: $A = A^{\frac{1}{2}}A^{\frac{1}{2}}$ with $A^{\frac{1}{2}}$ invertible. One can then deduce from Item 4 (and the fact that $A^* = A$ is invertible):

$$A \succeq I_n \quad \iff \quad A^{\frac{1}{2}}I_nA^{\frac{1}{2}} \succeq I_n \quad \iff \quad I_n \succeq A^{-\frac{1}{2}}I_nA^{-\frac{1}{2}} \quad \iff \quad I_n \succeq A^{-1}.$$

In the general case, if we assume $A \succeq B$, then Theorem 6.8, Item 4 and the upper result imply:

$$\begin{aligned} A - B = A^{\frac{1}{2}}(I_n - A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \succeq 0 \quad &\iff \quad I_n \succeq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \\ &\iff \quad A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} \succeq I_n \quad \iff \quad B^{-1} \succeq A^{-1}. \end{aligned}$$

6. Let us decompose $A = V^*\Lambda V$ with V unitary and Λ diagonal. We have $\Lambda \succeq V^{-*}I_nV^* \succeq I_n$ thanks to Item 4 thus $\lambda_i(A) - 1 \geq 0 \; \forall i \in [n]$.

□

Looking at the last result, one is tempted to ask what would happen if one replaces I_n with a matrix $B \in \mathcal{H}^n$. Denoting $\lambda_1(B) \leq \ldots \leq \lambda_n(B)$ the ordered list of the eigenvalues of B, do we have

$$A \succeq B \quad \implies \quad \lambda_i(A) \geq \lambda_i(B) \quad \forall i = 1, \ldots, m \; ?$$

The answer is yes but one will need supplementary results to provide a proof.

Theorem 6.10 (Rayleigh-Ritz)**.** *Given $A \in \mathcal{H}_n$ then*

$$\lambda_{\max}(A) = \max_{x \in \mathbb{C}^n} \frac{x^*Ax}{x^*x} \quad s.t. \; \|x\|_2 = 1$$

$$\lambda_{\min}(A) = \min_{x \in \mathbb{C}^n} \frac{x^*Ax}{x^*x} \quad s.t. \; \|x\|_2 = 1$$

[2] Recall that $P^{-*} = (P^*)^*$.

Proof. Let us decompose $A = V\Lambda V^*$ where $V = [v_1, \dots, v_n]$ is unitary and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \leq \cdots \leq \lambda_n$. First we know that:

$$\frac{v_n^* A v_n}{v_n^* v_n} = \lambda_n = \lambda_{\max}(A) \qquad \text{and} \qquad \frac{v_1^* A v_1}{v_1^* v_1} = \lambda_1 = \lambda_{\min}(A).$$

We now want to show that v_n and v_1 respectively maximize and minimize the functional $x \mapsto \frac{x^*Ax}{x^*x}$. It is merely sufficient to bound:

$$\begin{aligned} x^*Ax = x^*V\Lambda V^*x &= \sum_{i=1}^n \lambda_i |v_i^*x|^2 \\ &\leq \lambda_{\max}(A) \sum_{i=1}^n |v_i^*x|^2 = \lambda_{\max}(A) x^*VV^*x \quad = \lambda_{\max}(A)\|x\|_2^2, \end{aligned}$$

and one can check similarly that $\forall x \in \mathbb{C}^n$, $\lambda_{\min}(A)\|x\|_2^2 \leq x^*Ax$. □

Given a matrix $A \in \mathcal{H}_n$, we will note in what follows:

$$\lambda_1(A), \dots, \lambda_n(A), \qquad \text{with} \quad \lambda_1(A) \leq \cdots \leq \lambda_n(A),$$

the ordered list of eigenvalues of A.

Theorem 6.11 (Courant–Fischer)**.** *Let $A \in M_n$ be Hermitian and $k \in [n]$:*

$$\lambda_k = \min_{\dim S = k} \max_{x \in S^*} \frac{x^*Ax}{x^*x} \tag{6.1}$$

and

$$\lambda_k = \max_{\dim S = n-k+1} \min_{x \in S^*} \frac{x^*Ax}{x^*x}. \tag{6.2}$$

Proof. Let $x_1, \dots, x_n \in \mathbb{C}^n$ be orthonormal and such that $Ax_i = \lambda_i x_i$ for each $i = 1, \dots, n$. Let S be any k-dimensional subspace of $\mathbb{C}^n$ and let $S' = \operatorname{span}\{x_k, \dots, x_n\}$. Then

$$\dim S + \dim S' = k + (n-k+1) = n+1$$

therefore $\{x : 0 \neq x \in S \cap S'\}$ is nonempty. One can then bound:

$$\sup_{\substack{\|x\|=1 \\ x \in S}} \frac{x^*Ax}{x^*x} \geq \sup_{\substack{\|x\|=1 \\ x \in S \cap S'}} \frac{x^*Ax}{x^*x} \geq \inf_{\substack{\|x\|=1 \\ x \in S \cap S'}} \frac{x^*Ax}{x^*x} \geq \inf_{\substack{\|x\|=1 \\ x \in S'}} \frac{x^*Ax}{x^*x} = \min_{\substack{\|x\|=1 \\ x \in S'}} \frac{x^*Ax}{x^*x} = \lambda_k$$

which implies that

$$\inf_{\dim S = k} \sup_{\substack{x : \|x\|=1 \\ x \in S}} \frac{x^*Ax}{x^*x} \geq \lambda_k.$$

However, $\operatorname{span}\{x_1, \dots, x_k\}$ contains the eigenvector x_k, $\operatorname{span}\{x_1, \dots, x_k\}$ is one of the choices for the subspace S, and $\frac{x^*Ax}{x^*x} = \lambda_k$ when $x = x_k$, so the inequality above is actually an equality in which the infimum and supremum are reached:

$$\inf_{\dim S = k} \sup_{\substack{\|x\|=1 \\ x \in S}} \frac{x^*Ax}{x^*x} = \lambda_k$$

The second identity follows from applying the first result to $-A$:

$$-\lambda_k = \min_{\dim S = n-k+1} \max_{\substack{\|x\|=1 \\ x \notin S}} \frac{x^*(-A)x}{x^*x} \quad = \min_{\dim S = n-k+1} \max \left(-\frac{x^*Ax}{x^*x} \right) = -\left(\max_{\dim S = n-k+1} \min_{\substack{\|x\|=1 \\ x \notin S}} \frac{x^*Ax}{x^*x} \right)$$

□

Theorem 6.12 (Weyl). *Let $A, B \in \mathbb{H}^n$:*

$$\forall k \in [n]: \quad \lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_n(B).$$

Proof. Using the fact that $x^*Ax + x^*Bx = x^*(A+B)x$ and that for any $x \in \mathbb{C}^n \setminus \{0\}$:

$$\lambda_1(B) \leq \frac{x^*Bx}{\|x\|^2} \leq \lambda_n(B),$$

one has the inequality:

$$\frac{x^*Ax}{\|x\|^2} + \lambda_n(B) \leq \frac{x^*Ax}{\|x\|^2} + \frac{x^*Bx}{\|x\|^2} \leq \left(\frac{x^*Ax}{\|x\|^2}\right) + \lambda_n(B)$$

Composing on the left by $\min_{\dim(S)=k} \max_{\substack{x\in S\\ x\neq 0}}$, one obtains thanks to Theorem 6.11:

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \geq \lambda_k(A) + \lambda_n(B)$$

□

Corollary 6.13. *Given two Hermitian matrices $A, B \in \mathcal{H}_n$:*

$$A \succeq B \quad \Longrightarrow \quad \forall i \in [n]: \ \lambda_i(A) \geq \lambda_i(B),$$

where $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ (resp. $\lambda_1(B) \leq \cdots \leq \lambda_n(B)$) are the ordered list of eigenvalues of A (resp. B).

Proof. If $A \succeq B$, then $A - B \succeq 0$ and we have $\lambda_i(A-B) \geq 0$ for $i = 1, \ldots, n$ and therefore:

$$\lambda_i(A) = \lambda_i(A+B-B) \geq \lambda_i(B) + \lambda_1(A-B) \geq \lambda_i(B)$$

□

We will use a lot the following identity valid for a given $A \in \mathcal{H}_n$ and $i \in [n]$:

$$\lambda_i(A) = -\lambda_{n-i+1}(-A). \tag{6.3}$$

Proposition 6.14. *Let $A, B \in \mathcal{H}_n$, we have the following properties:*

1. *(Interlacing) Given any $z \in \mathbb{C}^n$, $k \in \{2, \ldots, n\}$:*

$$\lambda_{k-1}(A) \leq \lambda_k(A \pm zz^*) \leq \lambda_{k+1}(A)$$

2. *If rank$(B) \leq r$, $k \in \{r+1, \ldots, n-r\}$ we have:*

$$\lambda_{k-r}(A) \leq \lambda_k(A+B) \leq \lambda_{k+r}(A)$$

3. *For any index set $I = \{i_1, \ldots, i_r\} \subseteq \{1, 2, \ldots, n\}$,*

$$\lambda_{k-(n-r)}(A) \leq \lambda_k(A_I) \leq \lambda_{k+n-r}(A)$$

4. *For any semi-unitary matrix[3] $U \in \mathcal{M}_{n,r}(\mathbb{C})$,*

$$\lambda_{k-n+r}(A) \leq \lambda_k(U^*AU) \leq \lambda_k(A)$$

5. *Given $j, k \in [n]$:*

$$\lambda_{j+k-n}(A+B) \leq \lambda_j(A) + \lambda_k(B) \leq \lambda_{j+k-1}(A+B)$$

[3] A matrix with orthonormal columns (but possibly $r \neq n$ so it is not a square matrix).

Proof. 1. We know from Theorem 6.11 that:

$$\lambda_k(A \pm zz^*) = \min_{\dim S = k} \max_{x \in S} \frac{x^*(A \pm zz^*)x}{\|x\|^2} \geq \min_{\dim S = k} \max_{x \in S \cap \{z\}^\perp} \frac{x^*(A + zz^*)x}{\|x\|^2}$$

Now for all subspace $S \subset \mathbb{C}^n$, $\dim S = k$, there exists a subset $S^{\perp z} \subset S$ such that $\dim S^{\perp z} = k - 1$ and $S^{\perp z} \perp z$. Then:

$$\lambda_k(A \pm zz^*) \geq \min_{\dim S = k} \max_{x \in S^{\perp z}} \frac{x^*Ax}{\|x\|^2} = \min_{\substack{S' s.t. \exists S: \\ \dim(S) = k, S' = S^{\perp z}}} \max_{x \in S'} \frac{x^*Ax}{\|x\|^2} \geq \min_{\dim S = k-1} \max_{x \in S} \frac{x^*Ax}{\|x\|^2}$$

2. If $\mathrm{Rk}(B) \leq r$, given the eigenvalue decomposition $B = \sum_{i=1}^r \lambda_i(B) u_i u_i^*$, then:

$$\lambda_k(A + B) = \lambda_k\left(A + \sum_{i=1}^r \lambda_i(B) u_i u_i^*\right) \geq \lambda_{k-1}\left(A + \sum_{i=1}^{r-1} \lambda_i(B) u_i u_i^*\right) = \cdots = \lambda_{k-r}(A)$$

3. Recall that $A_I = (A_{i,j})_{i,j \in I} \in \mathcal{M}_r(\mathbb{C})$. We already know from Proposition 6.5 that A

Let us introduce the mapping $\phi_I : \mathbb{C}^r \mapsto \mathbb{C}^k n$ such that for all $x \in \mathbb{C}^r$, $\forall k \in [r]$, $\phi_I(x)_{i_k} = x_l$ and for all $i \in [n] \setminus I$, $\phi_I(x)_i = 0$. One can then express thanks to Courant Fischer Theorem that for any $k \in [r]$:

$$\begin{aligned}
\lambda_k(A_I) &= \min_{\substack{S \subset \mathbb{C}^r \\ \dim S = k}} \max_{x \in S, x \neq 0} \frac{x^* A_I x}{\|x\|^2} = \min_{\substack{S \subset \mathbb{C}^r \\ \dim S = k}} \max_{x \in S, x \neq 0} \frac{\phi_I(x)^* A \phi_I(x)}{\|\phi_I(x)\|^2} \\
&= \min_{\substack{S \subset \mathbb{C}^r \\ \dim S = k}} \max_{x \in \phi_I(S), x \neq 0} \frac{x^*Ax}{\|x\|^2} = \min_{\substack{S' \subset \mathbb{C}^n, \exists S \in \mathbb{C}^r: \\ \dim S = k, S' = \phi_I(S)}} \max_{x \in S', x \neq 0} \frac{x^*Ax}{\|x\|^2} \\
&\leq \min_{\substack{S' \subset \mathbb{C}^n, \\ \dim S = k+n-r}} \max_{x \in S', x \neq 0} \frac{x^*Ax}{\|x\|^2} \quad = \quad \lambda_{k+n-r}(A)
\end{aligned}$$

To prove the other equality let us note that for $l = n - k + 1$ and $A = -A'$, one has (multiplying the two sides of the inequality by (-1)):

$$-\lambda_{2n-l-r+1}(-A) \leq -\lambda_{n-l+1}(A_I),$$

which provides, thanks to (6.3):

$$\lambda_{l+r-n}(A) \leq \lambda_l(A_I)$$

4. Let us complete the orthonormal family induced by the r columns of $U = (u_1, \ldots, u_r)$ with $n-k$ vectors $(v_1, \ldots, v_{n-k})$. $W = (u_1, \ldots, u_r, v_1, \ldots, v_{n-r})$ unitary. Introducing the matrix $V = (v_1, \ldots, v_{n-k})$, one knows that the block matrix $\left(\begin{smallmatrix} U & V \end{smallmatrix} \right)$ is unitary and one can bound thanks to Item 3:

$$\lambda_{k+n-r}(A) = \lambda_{k+n-r}(W^*AW) = \lambda_{k+n-r}(\left(\begin{smallmatrix} U & V \end{smallmatrix} \right)^* A \left(\begin{smallmatrix} U & V \end{smallmatrix} \right)) = \lambda_{k+n-r}(\left(\begin{smallmatrix} U^*AU & U^*AV \\ V^*AU & V^*AV \end{smallmatrix} \right)) \geq \lambda_k(U^*AU),$$

thanks to Item 3. The other inequality is proven the same way thanks to Item 3.

5. Given $A, B \geq 0$, we have the eigenvalue decomposition

$$A = \sum_{i=1}^n \lambda_i(A) u_i(A) u_i(A)^* \qquad \text{and} \qquad B = \sum_{i=1}^n \lambda_i(B) u_i(B) u_i(B)^*.$$

Defining A_j as

$$A_j = \sum_{i=j+1}^n \lambda_i(A) u_i(A) u_i(A)^* \qquad \text{and} \qquad B_k = \sum_{i=k+1}^n \lambda_i(B) u_i(B) u_i(B)^*$$

we can show that

$$\lambda_j(A) = \lambda_n(A - A_j) \qquad \text{and} \qquad \lambda_k(B) = \lambda_n(B - B_k).$$

Now, since $\mathrm{Rk}(A_j - B_k) \leq 2n - j - k$ we know from Item 2:

$$\begin{aligned}\lambda_j(A) + \lambda_k(B) = \lambda_n(A - A_j) + \lambda_n(B - B_k) &\geq \lambda_n(A - A_j + B - B_k) \\ = \lambda_n(A + B - (A_j - B_k)) &\geq \lambda_{j+k-n}(A + B).\end{aligned}$$

One can deduce from the first result the sequence of implications and (6.3):

$$-\lambda_{j+k-n}(A+B) \geq -\lambda_j(A) - \lambda_k(B) \implies \lambda_{2n-j-k+1}(-A-B) \geq \lambda_{n-j+1}(-A) + \lambda_{n-k+1}(-B)$$

which implies that for all $A, B \in \mathcal{H}_n$ and any $j, k \in [n]$:

$$\lambda_{j+k-1}(A+B) \geq \lambda_j(A) + \lambda_k(B)$$

□

4 Applications

4.1 Principal component analysis (PCA)

Consider an $p \times n$ data matrix $X = (x_1, \ldots, x_n) \in \mathcal{M}_{p,n}$. Each of the n columns represents a different repetition of the experiment, and each of the p rows could be a particular kind of feature. PCA transforms the data into a new coordinate system through a linear transformation.

The transformation is defined by a family of l orthonormal weight vectors $(w_1, \ldots, w_l)$, and maps each column vector x_i of X to a new vector of principal component scores $t_i \in \mathbb{R}^l$, where l is generally less than p to reduce dimensionality. These scores are given for all $i \in [n]$ by:

$$t_i = W^T x_i \qquad \text{with :} \quad W \equiv (w_1, \ldots, w_l) \in \mathcal{M}_{p,l}(\mathbb{R}).$$

this way, one would go from a matrix X of size $p \times n$ to a matrix $T = (t_1, \ldots, t_n)$ of size $k \times n$. The score vectors $t_1, \ldots, t_n$ are seen as n drawings of the same law. One then construct each of the w_i with the objective to maximize the empirical variance of each of the entries of the score vector t_i. This condition rewrites for the first weight vector:

$$w_1 = \underset{\substack{w \in \mathbb{R}^p \\ \|w\|=1}}{\arg\max} \frac{1}{n} \sum_{i=1}^{n} (w^T x_i - w^T \bar{x})^2 \qquad \text{with:} \quad \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

with the assumption $\bar{x} = \mathbf{0}$ this identity rewrites:

$$w_1 = \underset{\substack{w \in \mathbb{R}^p \\ \|w\|=1}}{\arg\max} \frac{1}{n} \sum_{i=1}^{n} w^T x_i x_i^T w = \underset{\substack{w \in \mathbb{R}^p \\ \|w\|=1}}{\arg\max} \frac{1}{n} w^T X X^T w.$$

One recognizes here the Rayleigh quotient of the "sample covariance" $\frac{1}{n} X X^T$ and can deduce that $w_1 \in \mathbb{R}^p$ is the eigenvector associated to the biggest eigenvalue of $\frac{1}{n} X X^T \in \mathcal{M}_p$.

Subsequent components are found by subtracting the contribution of the previous components and finding the weight vector that extracts maximum variance from this new data matrix:

$$\forall k \in [l] : \quad W_k = \arg \max_{\|w\|=1} \left\{ \frac{w^\mathsf{T} \hat{X}_k^\mathsf{T} \hat{X}_k w}{w^\mathsf{T} w} \right\} \qquad \text{where :} \quad \hat{X}_k = X - \sum_{s=1}^{k-1} w_s w_s^\mathsf{T} X$$

The full PCA decomposition of X is then given by $T = XW$, where W is a matrix of weights whose columns are the eigenvectors associated to the l biggest eigenvalues of the sample covariance matrix $\frac{1}{n} X^\mathsf{T} X$.

4.2 Spectral analysis - MUSIC

Consider a complex time sequence

$$y_t = \sum_{l=1}^{k} \alpha_l \cdot e^{\mathbf{i}2\pi f_l t} + w_t, \quad t = 0, 1, \ldots, T-1$$

where $\forall l \in [k]$, $\alpha_l \in \mathbb{C}$ are complex coefficients and $f_l \in [0,1)$ are the frequencies of the sinusoidal components.

$$\textbf{Goal:} \quad \text{Estimate}\{f_l\}_{l=1}^{k} \quad \text{from} \quad \{y_t\}_{t=0}^{T-1}$$

This can be done by applying the Discrete Fourier Transform (DFT), but its resolution is limited by $\frac{1}{T}$. To achieve a super-resolution, a subspace-based approach is proposed.

Given a time window $d \geq k$, we define:

$$\forall t \in [T-d]: \quad Y_t^{(d)} = \begin{pmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+d-1} \end{pmatrix}, \; W_t^{(d)} = \begin{pmatrix} w_t \\ w_{t+1} \\ \vdots \\ w_{t+d-1} \end{pmatrix} \qquad \text{and} \qquad D = \mathrm{Diag}(\alpha_1, \ldots, \alpha_k)$$

.

For all $l \in [k]$, denote $z_l = e^{\mathbf{i}2\pi f_l}$, then given $d \geq k$

$$Y_t^{(d)} = \begin{pmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+d} \end{pmatrix} = \sum_{i=1}^{k} \alpha_i \begin{pmatrix} z_i^t \\ z_i^{t+1} \\ \vdots \\ z_i^{t+d} \end{pmatrix} + W_t^{(d)} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_k \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{d-1} & z_2^{d-1} & \cdots & z_k^{d-1} \end{pmatrix} \begin{pmatrix} \alpha_1 z_1^t \\ \alpha_2 z_2^t \\ \vdots \\ \alpha_k z_k^t \end{pmatrix} + W_t^{(d)}.$$

Given an exponent $p \in \mathbb{N}$ we introduce the Vandermonde matrix:

$$V_{(p)} \equiv \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_k \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{p-1} & z_2^{p-1} & \cdots & z_k^{p-1} \end{pmatrix}$$

Then one has the matricial identity $Y = V_{(d)} D V_{(T-D)}^* + W$ with:

$$Y = (Y_0^{(d)}, \ldots, Y_{T-d}^{(d)}) = \begin{pmatrix} y_t & \cdots & y_{t+T} \\ \vdots & & \vdots \\ y_{t+d-1} & \cdots & y_{t+T+d-1} \end{pmatrix}$$

$$W = (W_0^{(d)}, \ldots, W_{T-d}^{(d)}) = \begin{pmatrix} w_t & \cdots & w_{t+T} \\ \vdots & & \vdots \\ w_{t+d-1} & \cdots & w_{t+T+d-1} \end{pmatrix}.$$

Proposition 6.15. *If the scalars $z_1, \ldots, z_k$ are all distinct from one another then the Vandermonde matrix $V_{(p)} \in \mathcal{M}_{p,k}$, has a rank equal to* $\min\{p, k\}$.

Proof. To prove this, it's sufficient to consider an $l \times l$ submatrix of $V_{(p)}$ with $l \leq \min\{d, k\}$:

$$V_l \equiv \begin{pmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_l \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{l-1} & z_2^{l-1} & \ldots & z_l^{l-1} \end{pmatrix} \in \mathcal{M}_l(\mathbb{C})$$

Assume that we are given $x \in \mathbb{C}^l$ such that

$$\mathbf{0} = V_l^{\mathsf{T}} \cdot x = \begin{pmatrix} \sum_{i=1}^l x_i z_1^{i-1} \\ \vdots \\ \sum_{i=1}^l x_i z_l^{i-1} \end{pmatrix}.$$

This implies that the $z_1, \ldots, z_l$ are roots of a polynomial $p(z) = x_1 + x_2 z + \cdots + x_l z^{l-1}$ which has degree less than k. It means that this polynomial is equal to zero and therefore $x = \mathbf{0}$ and the the matrix V_l is injective thus invertible. $\square$

Denoting for simplicity $T_d \equiv T - d$, one can define an analogous to the sample covariance matrix (but here there is no independence between the columns) as:

$$R_y = \frac{1}{T_d} Y Y^* = \frac{1}{T_d} \left(V_{(d)} D V_{(T_d)}^* + W \right) \left(V_{(d)} D V_{(T_d)}^* + W \right)^* \approx \Phi + \frac{\sigma^2}{T_d} I_d \tag{6.4}$$

where σ^2 is the noise power, and $\Phi \equiv \frac{1}{T_d} V_{(d)} D V_{(T_d)}^* V_{(T_d)} D^* V_{(d)}^* \in \mathcal{M}_d(\mathbb{C})$ is a positive definite matrix of rank k.

Let us introduce the eigenvalue decomposition $R_y = U^* \Lambda U$, with U unitary and $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_d)$ diagonal with positive entries in increasing order $\lambda_1 \leq \cdots \leq \lambda_d$. Since $\mathrm{Rk}(\Phi) = k$, one can estimate:

$$\begin{aligned} \Phi \approx U^* \Lambda U - \frac{\sigma^2}{T_d} I_d &= \begin{pmatrix} U_n^* & U_s^* \end{pmatrix} \mathrm{Diag} \left(\underbrace{0, \ldots, 0}_{n-k \text{ times}}, \lambda_{d-k+1} - \frac{\sigma^2}{T_d}, \ldots, \lambda_n - \frac{\sigma^2}{T_d} \right) \begin{pmatrix} U_n \\ U_s \end{pmatrix} \\ &= U_s^* \mathrm{Diag} \left(\lambda_{d-k+1} - \frac{\sigma^2}{T_d}, \ldots, \lambda_n - \frac{\sigma^2}{T_d} \right) U_s, \end{aligned}$$

where we introduced the block decomposition $U = \left(\begin{smallmatrix} U_n \\ U_s \end{smallmatrix} \right)$ with $U_n \in \mathcal{M}_{d-k,d}$ and $U_s \in \mathcal{M}_{k,d}$. Of course, U being unitary, $U_s U_n^* = 0$ and therefore:

$$\frac{1}{T_d} U_n V_{(d)} D V_{(T_d)}^* V_{(T_d)} D^* V_{(d)}^* U_n^* = U_n \Phi U_n^* \approx U_n U_s^* \mathrm{Diag} \left(\lambda_{d-k+1} - \frac{\sigma^2}{T_d}, \ldots, \lambda_n - \frac{\sigma^2}{T_d} \right) U_s U_n^* = \mathbf{0}.$$

That means, since $D V_{(T_d)}^* V_{(T_d)} D^* \in \mathcal{M}_k(\mathbb{C})$ is of full rank, that:

$$0 \approx U_n V_{(d)} V_{(d)}^* U_n^* = \sum_{l=1}^k \| U_n^* a(f_l) \|^2,$$

where, $\forall l \in [k]$, $\forall f \in [0, 1)$, we introduced:

$$a(f) \equiv \begin{pmatrix} 1 \\ \vdots \\ z(f)^d \end{pmatrix} \in \mathbb{C}^d, \qquad \text{with} \quad z(f) \equiv e^{\mathrm{i} 2\pi f}.$$

In order to pick suited frequencies $f_1, \ldots, f_k$, one therefore needs to:

1. Choose appropriate $k \leq d$ (no precise method to optimally choose k),
2. Introduce $U_n \in \mathcal{M}_{d,n-k}$, a unitary matrix having as column an orthonormal basis of eigenvectors associated to the $n - k$ lowest eigenvalues of R_y,
3. Pick k frequencies such that $\| U_n a(f) \|$ is the closest possible to zero.

This last step is solved looking at the maximum of the mapping depicted on Figure 6.1.

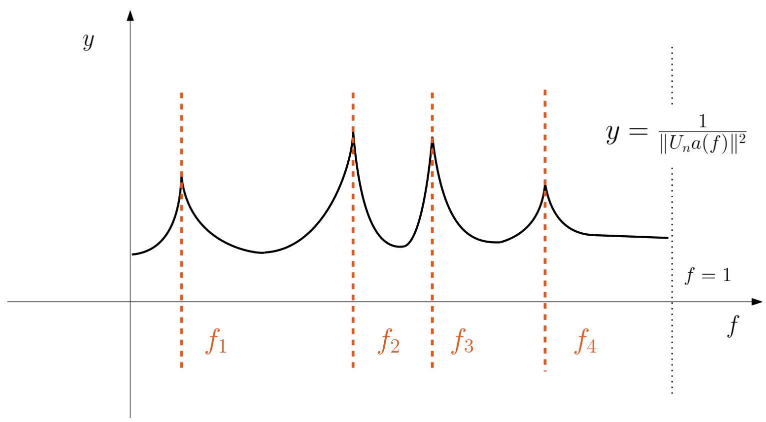


Figure 6.1: With this frequency selection graph, one is tempted to chose $\forall l \in [k]$, $z_l = e^{\mathrm{i}2\pi f_l}$.

Lecture 7

QR decomposition and applications.

1 Result and implementation

1.1 Existence and uniqueness

Theorem 7.1. *Let $A \in M_{n,m}$ be given.*

1. *If $n \geq m$, there is a $Q \in M_{n,m}$ with orthonormal columns and an upper triangular $R \in M_m$ with nonnegative main diagonal entries such that $A = QR$.*
2. *If rank $A = m$, then the factors Q and R in 1. are uniquely determined and the main diagonal entries of R are all positive.*
3. *If $m = n$, then the factor Q in 1. is unitary.*
4. *There is a unitary $Q \in M_n$ and an upper triangular $R \in M_{n,m}$ with nonnegative diagonal entries such that $A = QR$.*
5. *If A is real, then the factors Q and R in 1. 2. 3. 4. may be taken to be real.*

Only the second item will be proven, the uniqueness part relies on the following well known lemma.

Lemma 7.2. *The inverse of an invertible upper (resp. lower) triangular matrix is also upper (resp. lower) triangular.*

Proof. Let us consider $R \in \mathcal{M}_n(\mathbb{C})$, upper triangular. Denoting $D \in \mathcal{M}_n(\mathbb{C})$, the diagonal matrix having the same diagonal entries as R, one can decompose $R = (I_n + N)D$ where $N \in \mathcal{M}_n(\mathbb{C})$ satisfies $\forall j \leq i$, $N_{i,j} = 0$. One verifies easily that N is nilpotent, in particular $N^n = \mathbf{0}$. Let us then compute:

$$\begin{aligned} &RD^{-1}\left(I_n - N + N^2 - N^3 + \cdots + (-1)^{n-1}N^{n-1}\right) \\ &\qquad = (I_n + N)\left(I_n - N + N^2 - N^3 + \cdots + (-1)^{n-1}N^{n-1}\right) \\ &\qquad = I_n + N - N + N^2 - N^2 + \cdots + N^{n-1} - N^{n-1} + N^n \;\; = \;\; I_n, \end{aligned}$$

One can then conclude that $D^{-1}\left(I_n - N + \cdots + (-1)^{n-1}N^{n-1}\right)$ is the inverse of R, note that it is upper triangular as a sum of a product of upper triangular matrices. □

Proof of Theorem 7.1, Item 2. Let us note $x_1, \dots, x_m$ the columns of X, we know that those vectors form a linearly independent family of $\mathbb{C}^n$ since X is invertible. To show the existence of Q, R, let us invoke the Gram-Schmidt orthonormalizing process that ensures the existence of a orthonormal family $q_1, \dots, q_m$ $in\mathcal{C}^n$ such that for all $k \in [m]$:

$$x_k \in \{q_1, \dots, q_k\}.$$

In other words, for all $k \in [m]$, there exists k scalars $\alpha_{1,k}, \ldots, \alpha_{k,k}$ such that:

$$x_k = \sum_{i=1}^{k} \alpha_{i,k} q_i. \tag{7.1}$$

Introducing the unitary matrix $Q \equiv (q_1, \ldots, q_m)$ and the upper triangular matrix $R \in \mathcal{M}_m$ satisfying for all $j < i$, $R_{i,j} = 0$ and for all $j \geq i$, $R_{i,j} = \alpha_{i,j}$, (7.1) rewrites matricially $X = QR$. Then setting:

$$D = \mathrm{Diag}\left(\frac{R_{i,i}}{|R_{i,i}|}\right)_{i \in [m]},$$

(R as A is invertible, therefore, $\forall i \ in[n]$, $R_{i,i} \neq 0$), $Q' \equiv Q\bar{D}^{-1}$ and $R' \equiv \bar{D}R$, the decomposition $A = Q'R'$ satisfies the requirement of the decomposition.

Suppose that $A = QR = \tilde{Q}\tilde{R}$, in which R and $\tilde{R}$ are upper triangular and have positive main diagonal entries, and Q and $\tilde{Q}$ have orthonormal columns. Then $A^*A = R^*(Q^*Q)R = R^*IR = R^*R$ and also $A^*A = \tilde{R}^*\tilde{Q}^*\tilde{Q}\tilde{R} = \tilde{R}^*I\tilde{R} = \tilde{R}^*\tilde{R}$, so $R^*R = \tilde{R}^*\tilde{R}$ and consequently $\tilde{R}^{-*}R^* = \tilde{R}R^{-1}$. Thanks to thanks to Lemma 7.2, we know that $\tilde{R}^{-*}R^*$ is lower triangular and that $\tilde{R}R^{-1}$ is upper triangular. For a lower triangular matrix to equal an upper triangular matrix, both must be diagonal: $\tilde{R}R^{-1} = D$ is diagonal, and it must have positive main diagonal entries because the main diagonal entries of both $\tilde{R}$ and R^{-1} are positive. But $D = D^* = (\tilde{R}R^{-1})^* = R^{-*}\tilde{R}^* = (\tilde{R}^{-*}R^*)^{-1} = D^{-1}$, thus $D^2 = I_m$, hence $D = I_m$. We then conclude that $\tilde{R} = R$ and $\tilde{Q} = Q$.

□

There exist several method to compute the QR decomposition we will present three of them below.

1.2 Gram-Schmidt Procedure

Given $A = [a_1\ a_2\ \ldots\ a_n]$, for all $k \in [n]$, we want to estimate q_k and the scalars $r_{1,k}, r_{2,k}, \ldots, r_{k,k}$ satisfying:

$$a_k = \sum_{i=1}^{k} q_i r_{i,k} = q_1 r_{1,k} + q_2 r_{2,k} + \ldots + q_k r_{k,k}$$

They can be computed iteratively through the following steps:

- For $k = 1$, $a_1 = q_1 r_{11}$ thus set:

$$q_1 \equiv \frac{a_1}{\|a_1\|} \qquad \text{and} \qquad r_{11} \equiv \|a_1\|$$

- For $k = 2$, note that $r_{12} = q_1^* a_2$ and define:

$$y_2 \equiv a_2 - q_1 r_{12}, \qquad q_2 \equiv \frac{y_2}{\|y_2\|} \qquad \text{and} \qquad r_{22} \equiv q_2^* a_2$$

$$\vdots$$

- For k^{th} step, identity $a_k = q_1 r_{1,k} + \ldots + q_{k-1} r_{k-1,k}$ already imposes the choice $r_{i,k} \equiv q_i^* a_k$, then set:

$$y_k \equiv a_k - \sum_{i=1}^{k-1} r_{i,k} q_i, \qquad q_k \equiv \frac{y_k}{\|y_k\|} \qquad \text{and} \qquad r_{k,k} \equiv q_k^* a_k$$

1.3 Vectorized Gram-Schmidt Procedure

Let us note $r_1, \dots, r_n$, the columns of R^T, one has:

$$R = \begin{pmatrix} r_1^T \\ \vdots \\ r_n^T \end{pmatrix} \qquad \text{and therefore:} \quad A = \sum_{i=1}^n q_i r_i^T.$$

Note that for all $k \in \mathbb{N}$:

$$q_k^* \left(A - \sum_{i=1}^{k-1} q_i r_i^T \right) = \sum_{i=k}^n q_k^* q_i r_i^T = r_k^T.$$

We then follow the iterative computations:

- For $k = 1$:

$$q_1 \equiv \frac{a_1}{\|a_1\|} \qquad \text{and} \qquad r_1^T \equiv q_1^* A$$

$$\vdots$$

- For k^{th} step:

$$y_k \equiv \left(A - \sum_{i=1}^{k-1} q_i r_i^T \right) e_k, \qquad q_k \equiv \frac{y_k}{\|y_k\|} \qquad \text{and} \qquad r_k^T \equiv q_k^* A.$$

One can note with this iterative definition that of course:

$$QR = \sum_{i=1}^n q_i r_i^T = \sum_{i=1}^n q_i q_i^* A = A.$$

Moreover, given $l \leq k$:

$$q_l^* y_k = \sum_{i=k}^n q_l^* q_i r_i^T e_k = \begin{cases} 0 & \text{if } l < k \\ r_k^T e_k & \text{if } l = k, \end{cases}$$

thus the family of n vectors $q_1 = \frac{y_1}{\|y_1\|}, \dots, q_n = \frac{y_n}{\|y_n\|}$ is orthonormal by construction. Besides, for all $k \in [n]$, if we denote $A_k = \sum_{i=k}^n q_i r_i^T$, we see that $q_k = \frac{A_k e_k}{\|A_k e_k\|}$ and for all $i \in [k-1]$:

$$r_k^T = q_k^* A = q_k^* (\sum_{i=1}^n q_i r_i^T) = q_k^* (\sum_{i=i}^n q_i r_i^T) = q_k^* A_i,$$

Therefore for all $i < k$:

$$R_{k,i} = r_k^T e_i = q_k^* A_i e_i = q_k^* A_i e_i = q_k^* q_i = 0,$$

Thus the matrix R is upper triangular as expected.

Remark 7.3. *In practice, because of computation errors, at a step k, $q_1, \dots, q_k$ re not ideally orthogonal, to reduce the errors on the computation of r_k, it is then better to choose:*

$$r_k^T \equiv q_k^* \left(A - \sum_{i=1}^{k-1} q_i r_i^T \right).$$

1.4 Householder Transformation

Definition 7.1. *A projection matrix is a matrix $P \in \mathcal{M}_n(\mathbb{C})$ such that $P^2 = P$. A reflection matrix is a matrix of the form $H = I_n - 2P$ where $P \in \mathcal{M}_n(\mathbb{C})$ is a projection.*

Lemma 7.4. *A reflection matrix is an involution (it is its own inverse). If the associated projection is Hermitian then it is unitary.*

Proof. Simply note that for any projection P:

$$(I_n - 2P)^2 = I_n - 4P + 4P^2 = I_n.$$

Of course if $P^* = P$, then $H^* = I_n - 2P^* = H = H^{-1}$ and H is unitary. □

Lemma 7.5. *Given a vector $x \in \mathbb{C}^n$, different from e_1, let us introduce $\phi \in [0, 2\pi)$ such that $x^* e_1 = e^{\mathbf{i}\phi}|x^* e_1|$ and sets $v \equiv x - \|x\| e^{-\mathbf{i}\phi} e_1$. Denoting $P \equiv \frac{1}{\|v\|^2} vv^*$ and $H = I_n - 2P$, H is a reflexion and $Hx = e^{-\mathbf{i}\phi}\|x\| e_1$*

Proof. Note first that P is a projector (and consequently that H is a reflexion):

$$\forall y \in \mathbb{C}^n : \quad P^2 y = \frac{1}{\|v\|^4} vv^* vv^* y = \frac{1}{\|v\|^2} vv^* y = Py.$$

Besides, note that:

$$\|v\|^2 = \|x\|^2 - \|x\| e^{\mathbf{i}\phi} e_1^* x - \|x\| e^{-\mathbf{i}\phi} x^* e_1 + \|x\|^2 = 2\|x\|^2 - 2|x^* e_1| \|x\| = 2v^* x.$$

Thefore:

$$Hx = x - \frac{2vv^* x}{\|v\|^2} = x - v = \|x\| e^{-\mathbf{i}\phi} e_1.$$

□

The reflexion H is called a Householder transformation of x it is a hermitian reflexion (thus unitary see Lemma 7.4) depicted on it is depicted on Figure 7.1 in the case where $x^* e_1$ is real. We will then inroduce for all such $x \in \mathbb{C}^n$ $H(x) \equiv e^{\mathbf{i}\phi} H$, it is still a unitary matrix but not a reflexion nor a hermitian matrix anymore.

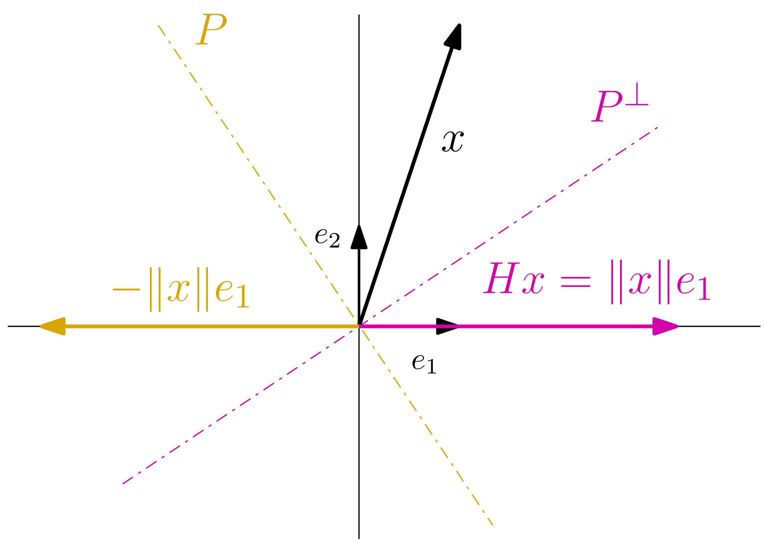


Figure 7.1: reflection of a vector x on $-\|x\|e_1$

To construct the QR decomposition of a given matrix A one can thus proceeds followingly:

- For $k = 1$, with the notations of Example 7.5, let us introduce $H_1 \equiv H(x_1)$ where x_1 is the first column of A. We know that $H_1 x_1 = \|x_1\| e_1$ and therefore:

$$H_1 A = \begin{bmatrix} \|x_1\| & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix} = \begin{bmatrix} \|x_1\| & * \\ 0 & A_2 \end{bmatrix},$$

  with $A_2 \in \mathcal{M}_{n-1}$

- For $k = 2$, let us introduce

$$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & H(x_2) \end{bmatrix},$$

  where $x_2 \in \mathbb{C}^{n-1}$ is the first column of $A_2 \in \mathcal{M}_{n-1}$. We have the identity:

$$H_2 H_1 A = \begin{bmatrix} 1 & 0 \\ 0 & H(x_2) \end{bmatrix} \begin{bmatrix} \|x_1\| & * \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} \|x_1\| & * & * \\ & \|x_2\| & * \\ (0) & & A_3 \end{bmatrix}$$

$$\vdots$$

Once we finished this iterative procedure, one just has to introduce the unitary matrix $Q = H_m \cdots H_2 H_1$ that satisfies $R \equiv QA$ is upper triangular, and therefore $A = Q^* R$ is the QR decomposition of A as described by Theorem 7.1.

2 Applications

2.1 Least square

Given $A \in \mathcal{M}_n$, $y \in \mathbb{C}^n$, we want to find $x \in \mathbb{C}^n$ solution to

$$x = \underset{x \in \mathbb{C}^n}{\arg\min} \|Ax - y\|^2 .$$

Let us assume that A admits the QR decomposition:

$$A = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \cdot \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1$$

with R_1 of full rank. One can then express:

$$\begin{aligned} \|Ax - y\|^2 = \|Q^*(Ax - y)\|^2 = \|Q^*(QRx - y)\|^2 &= \left\| \begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix} (Q_1 R_1 x - y) \right\|^2 \\ &= \left\| \begin{bmatrix} R_1 x - Q_1^* y \\ -Q_2^* y \end{bmatrix} \right\|^2 = \|R_1 x - Q_1^* y\|^2 + \|-Q_2^* y\|^2 \end{aligned}$$

The problem therefore boils down to minimizing $\|R_1 x - Q_1^* y\|^2$. The optimal vector $x \in \mathbb{C}^n$ cancels the gradient and therefore satisfies:

$$R_1^*(R_1 x - Q_1^* y) = 0 \iff R_1 x = Q_1^* y$$

When $R \in \mathcal{M}_n$ is upper triangular and $b \in \mathbb{C}^n$, the equation $Rx = b$ solves easily thanks to successive substitution ($x_n = \frac{b_n}{R_{n,n}}$, $x_{n-1} = \frac{b_{n-1} - R_{n-1,n-1} x_n}{R_{n-1,n}}$...).

2.2 Eigenvalue decomposition

The QR decomposition is widely used to compute eigenvalue decomposition of matrices. We provide below a simple algorithm to get such a decomposition. The justifications of the success of such an algorithm are quite elaborated, therefore we will not provide them in this lecture and simply mention that it rely on the same mechanism that makes the power method converge to the eigenvector associated to the biggest eigenvalue through successive iteration. Algorithm 2 will output a triangular matrix having the eigenvalues of A on the diagonal and a unitary matrix V whose first column is the eigenvector associated to the biggest eigenvalue of A. As such, this algorithm is a good alternative to the power method.

We provide below a lemma setting that the first output of the algorithm has the same eigenvalues as the input (the convergence of the algorithm and the fact that this output is upper triangular is not justified here).

Lemma 7.6. *Given a full rank matrix $A \in \mathcal{M}_n(\mathbb{C})$, and a sequence of scalars $(z_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, consider the sequences of matrices $(A_k)_{k\in\mathbb{N}} \in \mathcal{M}_n(\mathbb{C})^{\mathbb{N}}$ such that $A_0 = A$ and for all $k \geq 1$:*

$$A_k = R_{k-1}Q_{k-1} - z_k I_n,$$

where Q_k, R_k are respectively the unitary and upper triangular matrix resulting from the QR decomposition of $A_k + z_k I_n$. For all $k \in \mathbb{N}$, A and A_k are unitary similar

Proof. This result is merely proven iteratively thanks to the identity:

$$\begin{aligned} A_k &= R_{k-1}Q_{k-1} - z_k I_n = Q_{k-1}^*(A_{k-1} + z_{k-1}I_n)Q_{k-1} - z_k I_n = Q_{k-1}^* A_{k-1} Q_{k-1} \\ &= Q_{k-1}^* Q_{k-2}^* A_{k-2} Q_{k-2} Q_{k-1} = \cdots = Q_{k-1}^* \cdots Q_0^* A Q_0 \cdots Q_{k-1}, \end{aligned}$$

and, of course, the matrix $Q_0 \cdots Q_{k-1}$ is unitary as a product of unitary matrices. ☐

Algorithm 2 QR method.

```
Consider an initial guess x,
error = 1
A_k := A
V := I_n
while error > tol do
    A_aux := A_k.
    Draw ε ~ N(0,1) + iN(0,1).
    Q_k, R_k := QR decomposition of A_k + εI_n.
    A_k := R_k Q_k − εI_n.
    V = V Q_k
    error = ‖Sort(Diag(A_k)) − Sort(Diag(A_aux))‖
Output A_k, V.
```

In the description of the algorithm the shift ε is introduced to speed up the computations. In particular, when dealing with real matrices with complex eigenvalues, it is necessary to allow the convergence to a triangular matrix with complex diagonal entries as expected.

2.3 Canonical correlation analysis (CCA).

As PCA, CCA is used for dimension reduction but it takes as input two matrices instead of one. The main goal is generally to see how similar are two sets of data and provides some orthogonal projection on which this similarity can be revealed. Considering two data matrices $A \in \mathcal{M}_{m,n}(\mathbb{C})$ and $B \in \mathcal{M}_{m,e}(\mathbb{C})$. We want to find two low-rank approximations for A and B respectively that are close to each other.

We will see in next lecture a variational characterization of singular values (for non Hermitian matrices, otherwise, one could use the Rayleigh-Ritz theorem):

Given $A \in \mathcal{M}_{m,n}(\mathbb{C})$ the singular value decomposition can be defined with the following iteration:

$$\sigma_1(A) = \max_{x,y} \frac{|y^* Ax|}{\|y\|\|x\|} \equiv \frac{|u_1^* A v_1|}{\|u_1\|\|v_1\|} \qquad \forall k \geq 2: \quad \sigma_k(A) = \max_{\substack{x \perp u_1,\dots,u_{k-1} \\ y \perp v_1,\dots,v_{k-1}}} \frac{|y^* Ax|}{\|y\|\|x\|} \equiv \frac{|u_k^* A v_k|}{\|u_k\|\|v_k\|},$$

then $A = \sum_{i=1}^p \sigma_k(A) u_k v_k^*$ and $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_p(A) \geq 0$ with $p \equiv \mathrm{Rk}(A)$.

The CCA expresses in a similar way.

Definition 7.2 (Canonical correlation coefficient)**.** *Let $A \in \mathcal{M}_{m,n}(\mathbb{C})$, $B \in \mathcal{M}_{m,r}(\mathbb{C})$ and assume $p = rank(A) \geq rank(B) = q$. The canonical correlation coefficients, $\sigma_1(A,B), \ldots, \sigma_q(A,B)$ of the pair (A,B) are recursively defined as*

$$\sigma_1(A,B) = \max_{x,y} \frac{|y^* B^* Ax|}{\|By\|\|Ax\|} \equiv \frac{|y_1^* B^* A x_1|}{\|By_1\|\|Ax_1\|}$$
$$\forall k \in \{2,\ldots,q\}: \quad \sigma_k(A) = \max_{\substack{Ax \perp Ax_1,\dots,Ax_{k-1} \\ By \perp By_1,\dots,By_{k-1}}} \frac{|y^* B^* Ax|}{\|By\|\|Ax\|} \equiv \frac{|y_k^* B^* A x_k|}{\|By_k\|\|Ax_k\|},$$

Then the normalized vectors $\frac{Ax_k}{\|Ax_k\|}$, $\frac{By_k}{\|By_k\|}$, for $k = 1,\ldots,q$ are called canonical vectors.

If $p = q$, then $\{Ax^1, \ldots, Ax^q\}$ and $\{By^1, \ldots, By^q\}$ are orthonormal basis for $\mathrm{Im}(A)$ and $\mathrm{Im}(B)$ respectively. One also has the following bound on the canonical coefficients:

Lemma 7.7. *The canonical correlation coefficients of a pair of two matrices are always lower than* 1.

Proof. In the setting of Definition 7.2, one can bound thanks to Cauchy-Shwarz inequality:

$$\sigma_k(A) = \max_{\substack{Ax \perp Ax_1,\dots,Ax_{k-1} \\ By \perp By_1,\dots,By_{k-1}}} \frac{|y^* B^* Ax|}{\|By\|\|Ax\|} \leq 1.$$

□

Consider a QR decomposition of A and B:

$$A = Q_A R_A \qquad \text{and} \qquad B = Q_B R_B,$$

with $Q_A \in \mathcal{M}_{m,n}, Q_B \in \mathcal{M}_{m,r}$ semi-unitary and $R_A \in \mathcal{M}_n, R_B \in \mathcal{M}_r$ upper triangular invertible. Then:

$$\sigma_k(A,B) = \max_{\substack{Ax \perp Ax_1,\dots,Ax_{k-1} \\ By \perp By_1,\dots,By_{k-1}}} \frac{|y^* B^* Ax|}{\|By\|\|Ax\|} = \max_{\substack{Q_A R_A x \perp Q_A R_A x_1,\dots,Q_A R_A x_{k-1} \\ Q_B R_B y \perp Q_B R_B y_1,\dots,Q_B R_B y_{k-1}}} \frac{|y^* R_B^* Q_B^* Q_A R_A x|}{\|Q_B R_B y\|\|Q_A R_A x\|}$$
$$= \max_{\substack{x \perp x_1,\dots,x_{k-1} \\ y \perp y_1,\dots,y_{k-1}}} \frac{|y^* Q_B^* Q_A x|}{\|y\|\|x\|} = \sigma_k(Q_B^* Q_A),$$

the k^{th} singular value of $Q_B^* Q_A$. Note that if $Q_A = Q_B$ then $Q_B^* Q_A = I_m$ and all the canonical correlation coefficients are equal to 1.

Noting that $\sigma_k(A,B) = y_k^* B^* A x_k$ with $\|Ax_k\| = \|By_k\| = 1$ setting $X \equiv (x_1,\ldots,x_k)$ and $Y \equiv (y_1,\ldots,y_k)$, one has the equivalent formulations of the canonical correlation problem:

1. Find $X \in \mathcal{M}_{n,q}, Y \in \mathcal{M}_{n,q}$ that maximize $\mathrm{Tr}(Y^* B^* A X)$, s.t. $X^* A^* A X = Y^* B^* B Y = I_q$
2. Find $X \in \mathcal{M}_{n,q}, Y \in \mathcal{M}_{n,q}$ that minimize $\|BY - AX\|_F$, s.t. $X^* A^* A X = Y^* B^* B Y = I_q$.

Lecture 8

Singular values decomposition and applications

1 General results

Theorem 8.1 (Singular Value Decomposition). *Let $A \in \mathcal{M}_{n,m}(\mathbb{C})$ with $n \geq m$ be given. Then there exist unitary matrices $V \in \mathcal{M}_n(\mathbb{C})$ and $W \in \mathcal{M}_m(\mathbb{C})$ such that $A = V\Sigma W^*$ with $\Sigma = \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_{n,m}(\mathbb{R})$, $\Gamma = Diag(\sigma_1, \dots, \sigma_r)$, where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ and $r = Rk(A)$.*

A decomposition of the form specified in Theorem 8.1 is called a singular value decomposition (SVD) of the matrix A. The diagonal entries of the matrix Γ are called singular values and the columns of V and W are called left and right singular vectors of A.

Proof. If $A = 0$, then we set $V = I_n, \Sigma = 0 \in \mathcal{M}_{n,m}(\mathbb{C}), \Gamma = [I], W = I_m$, and we are finished.

If $A \neq 0$ and $r = \mathrm{Rk}(A)$, since $n \geq m$, we have $1 \leq r \leq m$, and since $A^*A \in \mathcal{M}_m(\mathbb{C})$ is Hermitian, there exists a unitary matrix $W = [w_1, \dots, w_m] \in \mathcal{M}_m(\mathbb{C})$ such that $W^*(A^*A)W = \mathrm{Diag}(\lambda_1, \dots, \lambda_m) \in \mathcal{M}_m(\mathbb{R})$.

Without loss of generality, we assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0$. Given $x \in \mathbb{C}^n$, one has the implications $Ax = 0 \implies A^*Ax = 0$ and $A^*Ax = 0 \implies \|Ax\| = x^*A^*Ax = 0 \implies Ax = 0$, thus $\mathrm{Ker}(A) = \mathrm{Ker}(A^*A)$ and consequently $\mathrm{Rk}(A) = \mathrm{Rk}(A^*A)$. Therefore, the matrix A^*A has exactly r positive eigenvalues $\lambda_1, \dots, \lambda_r$ and $m - r$ times the eigenvalue 0. Introducing the diagonal matrix $\Gamma \equiv \mathrm{Diag}(\lambda_1^{\frac{1}{2}}, \dots, \lambda_r^{\frac{1}{2}}) \in \mathcal{M}_r(\mathbb{R})$, let us further denote:

$$\Lambda \equiv \begin{bmatrix} \Gamma & 0 \\ 0 & I_{m-r} \end{bmatrix} \in \mathcal{M}_m(\mathbb{R}) \qquad \text{and} \qquad X = (x_1, \dots, x_m) \equiv AW\Lambda^{-1} \in \mathcal{M}_{n,m}.$$

If one introduce the decomposition $V \equiv (x_1, \dots, x_r)$ and $Z \equiv (x_{r+1}, \dots, x_m)$ than one can express:

$$\begin{bmatrix} V^*V & V^*Z \\ Z^*V & Z^*Z \end{bmatrix} = \begin{bmatrix} V^* \\ Z^* \end{bmatrix} \begin{bmatrix} V & Z \end{bmatrix} = XX^* = \Lambda^{-1}W^*A^*AW\Lambda^{-1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

which implies in particular that $Z = 0$ and $V^*V = I_r$. Let us then complete $(x_1, \dots, x_r)$ to create an orthonormal basis of $\mathbb{C}^n$: $(x_1, \dots, x_r, \tilde{x}_{r+1}, \dots, x_n)$. Then the matrix $U \equiv (V, \tilde{Z}) \in \mathcal{M}_n(\mathbb{C})$ is unitary, with $\tilde{Z} \equiv (\tilde{x}_{r+1}, \dots, x_n) \in \mathcal{M}_{n,n-r}$. One can then conclude:

$$A = \begin{bmatrix} V & 0 \end{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & I_{m-r} \end{bmatrix} W^* = \begin{bmatrix} V & \tilde{Z} \end{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} W^* = U\Sigma W^*.$$

□

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Lemma 8.2. *Suppose that the matrix $A \in \mathcal{M}_{n,m}(\mathbb{C})$ with rank$(A) = r$ has an SVD as specified in Theorem 8.1 with $V = [v_1, \ldots, v_n]$ and $W = [w_1, \ldots, w_m]$. We then have Im(A) = Span$\{v_1, \ldots, v_r\}$ and Ker(A) = Span$\{w_{r+1}, \ldots, w_m\}$.*

Proof. For $j = 1, \ldots, r$, we have $Aw_j = V\Sigma W^* w_j = V\Sigma e_j = \sigma_j v_j \neq 0$, since $\sigma_j \neq 0$. Hence, these r linearly independent vectors satisfy $v_1, \ldots, v_r \in \mathrm{Im}(A)$. Now $r = \mathrm{Rk}(A) = \dim(\mathrm{Im}(A))$ implies that $\mathrm{Im}(A) = \mathrm{Span}\{v_1, \ldots, v_r\}$.

For $j = r+1, \ldots, m$, we have $Aw_j = 0$, and hence these $m - r$ linearly independent vectors satisfy $w_{r+1}, \ldots, w_m \in \mathrm{Ker}(A)$. Then $\dim(\mathrm{Ker}(A)) = m - \dim(\mathrm{Im}(A)) = m - r$ implies that $\mathrm{Ker}(A) = \mathrm{Span}\{w_{r+1}, \ldots, w_m\}$. □

We end this presentation of the general results with an adaptation of Courant Fischer Theorem to singular values. The following result is given without proof since one simply has to apply Courant Fischer theorem to the matrix $(A^*A)^{\frac{1}{2}}$

Theorem 8.3 (Courant–Fischer for SVD)**.** *Let $A \in M_{n,m}$, $q = \min(n, m)$, we denote $0 \leq \sigma_1(A) \leq \cdots \leq \sigma_q(A)$, the q singular values of A. For any $k \in [q]$:*

$$\sigma_k(A) = \min_{\dim S = k} \max_{x \in S^*} \frac{\|Ax\|}{\|x\|} \tag{8.1}$$

and

$$\sigma_k(A) = \max_{\dim S = m-k+1} \min_{x \in S^*} \frac{\|Ax\|}{\|x\|}. \tag{8.2}$$

2 Applications

2.1 Low rank approximation

An SVD of the form $A = \sum_{j=1}^{r} \sigma_j v_j w_j^*$ can be written as a sum of r matrices of the form $\sigma_j v_j w_j^*$, where $\mathrm{Rk}(\sigma_j v_j w_j^*) = 1$. Let

$$A_k := \sum_{j=1}^{k} \sigma_j v_j w_j^* \text{ for some } k, 1 \leq k \leq r. \tag{8.3}$$

Then $\mathrm{Rk}(A_k) = k$ and, using that the matrix spectral norm[1] (or matrix 2-norm) $\|\cdot\|$ is unitarily invariant, we get

$$\|A - A_k\| = \|V^*(A - A_k)W\| = \|\mathrm{Diag}(\sigma_{k+1}, \ldots, \sigma_r)\| = \sigma_{k+1}.$$

Hence A is approximated by the matrix A_k, where the rank of the approximating matrix and the approximation error in the spectral norm are explicitly known. The singular value decomposition furthermore yields the best possible approximation of A by a matrix of rank k with respect to the spectral norm.

Theorem 8.4 (Optimal Approximation by SVD)**.** *With A_k as in (8.3), we have $\|A - A_k\| \leq \|A - B\|$ for every matrix B in $\mathcal{M}_{n,m}(\mathbb{C})$ with Rk$(B) = k$.*

Proof. The assertion is clear for $k = \mathrm{Rk}(A)$, since then $A_k = A$ and $\|A - A_k\| = 0$. Let $k < \mathrm{Rk}(A) \leq m$. Let $B \in \mathcal{M}_{n,m}(\mathbb{C})$ with $\mathrm{Rk}(B) = k$ be given, then $\dim(\mathrm{Ker}(B)) = m - k$. If $w_1, \ldots, w_m$ are the right singular vectors of A from the SVD, then $U := \mathrm{Span}\{w_1, \ldots, w_{k+1}\}$ has dimension $k + 1$. Since $\mathrm{Ker}(B)$ and U are subspaces of $\mathbb{C}^m$ with $\dim(\mathrm{Ker}(B)) + \dim(U) = m + 1$, we have $\mathrm{Ker}(B) \cap U \neq \{0\}$.

Let $v \in \mathrm{Ker}(B) \cap U$ with $\|v\| = 1$ be given. Then there exist $\alpha_1, \ldots, \alpha_{k+1}$ in $\mathbb{C}$ with $v = \sum_{j=1}^{k+1} \alpha_j w_j$ and $\sum_{j=1}^{k+1} |\alpha_j|^2 = \|v\|^2 = 1$. Hence

$$(A - B)v = Av - Bv = \sum_{j=1}^{k+1} \alpha_j \sigma_j v_j,$$

[1] $\forall A \in \mathcal{M}_{p,n}$, $\|A\| = \sup_{x \in \mathbb{C}^n \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$

and, therefore taking advantage of the fact that $(v_1, \dots, v_{k+1})$ is an orthonormal family:

$$\begin{aligned}\|A-B\|^2 = \max_{\|y\|=1} \|(A-B)y\|^2 \geq \|(A-B)v\|^2 = \left\|\sum_{j=1}^{k+1} \alpha_j \sigma_j v_j\right\|^2 = \sum_{j=1}^{k+1} |\alpha_j \sigma_j|^2 \\ \geq \sigma_{k+1}^2 \sum_{j=1}^{k+1} |\alpha_j|^2 = \sigma_{k+1}^2 = \|A - A_k\|^2,\end{aligned}$$

which completes the proof. □

2.2 Pseudo inverse

Another important application of the SVD arises in the solution of linear systems of equations. If $A \in \mathcal{M}_{n,m}(\mathbb{C})$ has an SVD of the form as given in Theorem 8.1, we define the matrix $A^\dagger$ as follows:

$$A^\dagger := W\Sigma^\dagger V^* \in \mathcal{M}_{m,n}(\mathbb{C}), \text{ where } \Sigma^\dagger := \begin{bmatrix} \Gamma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{m,n}.$$

One easily sees that $A^\dagger A = W \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} W^* \in \mathcal{M}_m(\mathbb{R})$. If $r = m = n$, then A is invertible and the right-hand side of the above equation is equal to the identity matrix I_n. In this case, we have $A^\dagger = A^{-1}$. The matrix $A^\dagger$ can therefore be viewed as a generalized inverse, that in the case of an invertible matrix A is equal to the inverse of A.

Definition 8.1. *The matrix $A^\dagger$ in is called Moore-Penrose inverse or pseudo-inverse of A.*

Let $A \in \mathcal{M}_{n,m}(\mathbb{C})$ and $b \in \mathbb{C}^n$ be given. If the linear system of equations $Ax = b$ has no solution, then we can try to find an $\hat{x}$ in $\mathbb{C}^m$ such that $A\hat{x}$ is "as close as possible" to b. Using the Moore-Penrose inverse we obtain the best possible approximation with respect to the Euclidean norm.

Theorem 8.5. *Let A be as given, with an SVD $A = V\Sigma W^*$ as in Theorem 8.1 and $A^\dagger$ the Moore-Penrose pseudo inverse defined in Definition 8.1, then $\|b - AA^\dagger b\| \leq \|b - Ay\|$ for all $y \in \mathbb{C}^m$, and the norm of $A^\dagger b$ is given by*

$$\|A^\dagger b\| = \left(\sum_{j=1}^{r} \left|\frac{v_j^* b}{\sigma_j}\right|^2\right)^{1/2} \leq \|y\|$$

for all y in $\mathbb{C}^m$ with $\|b - AA^\dagger b\| = \|b - Ay\|$.

Proof. Let $y \in \mathbb{C}^m$ be given and introduce the scalars $\xi_1, \dots, \xi_m \in \mathbb{C}$ such that $W^* y = (\xi_1, \dots, \xi_m)$. Then:

$$\|b - Ay\|^2 = \|V(V^*b - \Sigma z)\|^2 = \|V^*b - \Sigma z\|^2 = \sum_{j=1}^{r} |v_j^* b - \sigma_j \xi_j|^2 + \sum_{j=r+1}^{n} |v_j^* b|^2 \tag{8.4}$$

Now, noting that $b = \sum_{i=1}^n (v_i^* b) v_i$ and $AA^\dagger b = V\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)V^* b = \sum_{i=1}^r (v_i^* b) v_i$, one can also bound:

$$\|b - AA^\dagger b\|^2 = \left\|\sum_{i=r+1}^{n} (v_i^* b) v_i\right\|^2 = \sum_{i=r+1}^{n} |v_i^* b|^2 \leq \|b - Ay\|^2,$$

for any $y \in \mathbb{C}^m$.

One deduce from (8.4) that every vector $y \in \mathbb{C}^m$ that satisfies $\|b - Ay\| = \|b - AA^\dagger b\|$ must have the form $y = W(\frac{v_1^* b}{\sigma_1}, \dots, \frac{v_r^* b}{\sigma_r}, y_{r+1}, \dots, y_m)$ for some $y_{r+1}, \dots, y_m \in \mathbb{C}$. Recalling that $A^\dagger b = \sum_{i=1}^r \frac{v_i^* b}{\sigma_i} w_i$ that implies that:

$$\|y\|^2 = \|W^* y\|^2 = \sum_{i=1}^{r} \frac{|v_i^* b|^2}{\sigma_i^2} + \sum_{i=r+1}^{m} |y_i|^2 = \|A^\dagger b\|^2 + \sum_{i=r+1}^{m} |y_i|^2 \geq \|A^\dagger b\|^2.$$

□

Lecture 9

Triangular factorizations and canonical forms

If a linear system $Ax = b$ has a nonsingular triangular coefficient matrix $A \in M_n$, computation of the unique solution x is remarkably easy. If, for example, $A = (A_{ij})i, j \in [n]$ is upper triangular and nonsingular, then all $A_{ii} \neq 0$ and one can employ back substitution: $A_{nn}x_n = b_n$ determines x_n; $A_{n-1,n-1}x_{n-1} + A_{n-1,n}x_n = b_{n-1}$ then determines x_{n-1} since x_n is known and $A_{n-1,n-1} \neq 0$; proceeding in the same fashion upward through successive rows of A, one determines $x_{n-2}, x_{n-3}, \dots, x_2, x_1$.

If $A \in M_n$ is not triangular, one can still use forward and back substitution to solve $Ax = b$ provided that A is nonsingular and can be factored as $A = LU$, in which L is lower triangular and U is upper triangular: (i) use forward substitution to solve $Ly = b$, and (ii) use back substitution to solve $Ux = y$.

Given $A \in M_n$, an "LU factorization of A" is any decomposition $A = LU$, in which $L \in M_n$ is lower triangular and $U \in M_n$ is upper triangular.

Remark 9.1. *Let $A \in M_n$ and suppose that $A = LU$ is an LU factorization. For any block 2×2 partition*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix},$$

with $A_{11}, L_{11}, U_{11} \in M_k$ and $k < n$, we have:

$$\begin{cases} A_{11} = L_{11}U_{11} \\ A_{12} = L_{11}U_{12} \\ A_{21} = L_{21}U_{11} \\ A_{11} = L_{21}U_{12} + L_{22}U_{22}. \end{cases} \tag{9.1}$$

In particular, note that each leading principal submatrix of A (A_{11} in this example) has an LU factorization in which the factors are the corresponding leading submatrices of L and U.

Given $A \in \mathcal{M}_n$ and $i \in [n]$, recall the notation $A_{[i]} \equiv (A_{k,l})_{k,l \in [i]} \in \mathcal{M}_i$, the i^{th} principal submatrix of A. With these notations note that the matrices A_{11}, L_{11} and U_{11} can respectively be noted $A_{[k]}$, $L_{[k]}$ and $U_{[k]}$

Theorem 9.2. *Let $A \in M_n$ be given. Then*

1. *A has an LU factorization in which L is nonsingular if and only if A has the row inclusion property: For each $i = 1, \dots, n-1$, $(A_{i+1,j})_{j \in [i]}$ is a linear combination of the rows of $A_{[i]}$*

2. *A has an LU factorization in which U is nonsingular if and only if A has the column inclusion property: For each $j = 1, \dots, n-1$, $(A_{i,j+1})_{i \in [j]}$ is a linear combination of the columns of $A_{[j]}$*

This lecture is a close copy of the book of Roger A. Horn and Charles R. Johnson : Matrix Analysis

Proof. If $A = LU$, then as explained in Remark 9.1, $A_{[i+1]}$ also admits a LU factorization, more precisely $A_{[i+1]} = L_{[i+1]}U_{[i+1]}$. Thus, to verify the necessity of the row inclusion property, it suffices to take $i = k = n-1$ in the partitioned presentation given in Remark 9.1. Since L is nonsingular and triangular, L_{11} is also nonsingular, and we have $A_{21} = L_{21}U_{11} = L_{21}L_{11}^{-1}L_{11}U_{11} = (L_{21}L_{11}^{-1})A_{11}$, which verifies the row inclusion property.

Conversely, if A has the row inclusion property, we may construct inductively an LU factorization with nonsingular L as follows. The cases $n = 1, 2$ are easily verified. Given $k\ in[n]$, we then introduce the block decomposition $A_{[k]} = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ with $A_{11} = A_{[k-1]} \in \mathcal{M}_{k-1}$, $A_{21}^T, A_{12} \in \mathcal{M}_{k-1,1}$ and $A_{22} \in \mathcal{M}_1$. We further assume that $A_{11} = L_{11}U_{11}$ with L_{11} nonsingular, and that the row vector A_{21} is a linear combination of the rows of A_{11}. Then there is a vector y such that $A_{21} = y^T A_{11} = y^T L_{11}U_{11}$. Inspiring from the expressions provided in (9.1), note that choosing $L_{21} = y^T L_{11}$ ensures $A_{21} = y^T L_{11}U_{11} = L_{21}U_{11}$, we beside may take $U_{12} = L_{11}^{-1}A_{12}$, $L_{22} = 1$, and $U_{22} = A_{22} - L_{21}U_{12}$ to obtain an LU factorization:

$$A_{[k]} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \cdot \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix},$$

in which $L \equiv \left(\begin{smallmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{smallmatrix}\right)$ is non singular (since L_{11} is non singular and $L_{22} \neq 0$. One can then follow this procedure until $k = n$. □

Corollary 9.3. *(LDU factorization). Let $A = [a_{ij}] \in M_n$ be an invertible matrix. The matrix A has an LU factorization $A = LU$ if and only if $A_{[i]}$ is nonsingular for all $i = 1, \ldots, n$.*

Proof.
1. One may simply note that if all the principal submatrices are invertible then their columns or rows span the entire space. More precisely, given $k \in [n]$, any column of $(A_{i,j})_{i\in[k],j\in[n]}$ (resp. any row of $(A_{i,j})_{i\in[n],j\in[k]}$) is a linear combination of the columns (resp. of the row) of $A_{[k]}$. Conversely, since A is non singular, $A = LU$ implies that L and U are non singular. Since U and L are triangular, that implies in particular that $L_{[k]}$ and $U_{[k]}$ are non singular for all $k \in [n]$ and consequently that $A_{[k]} = L_{[k]}U_{[k]}$ is non singular.
2. *skipped.*

□

Example 9.4. *Not every matrix has an LU factorization. If*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

could be written as $A = LU = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$, *then $l_{11}u_{11} = 0$ implies that either $l_{11} = 0$ or $l_{11} = 0$. In the former case, that would imply that $1 = l_{11}u_{12} = 0$ and the latter case would imply that $1 = l_{21}u_{11} = 0$ which is also absurd.*

Lemma 9.5. *Let $A \in M_n$ be nonsingular. Then there is a permutation matrix P such that A can be factored as $A = PLU$ with L and U being lower and upper triangular matrices, respectively.*

Proof. The proof is by induction on n. If $n = 1$ or 2, the result is clear by inspection. Assume the result holds for $n-1$. Consider a nonsingular matrix $A \in M_n$. We know that the $(n-1)$ first columns are linearly independent, therefore there exist $n-1$ linearly independent rows of $(A_{i,j})_{i\in[n],j\in[n-1]}$ that can be put in first position through a permutation R. Then noting $B = RA$, we know that $B_{[n-1]}$ is non singular and the induction hypothesis allows us to introduce a permutation matrix $Q \in \mathcal{M}_{n-1}$ such that $B_{[n-1]} = QL'U'$ with L' lower triangular and U' upper triangular. We know from Theorem 9.2 that $Q^{-1}B_{n-1}$ satisfies the inclusion row property. Then by construction, setting, $\tilde{Q} \equiv \left(\begin{smallmatrix} Q^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we deduce that $\tilde{Q}B$ also satisfies the inclusion row property and that there exists L lower triangular and U upper triangular such that $\tilde{Q}B = LU$, setting $P \equiv R^{-1}\tilde{Q}^{-1}$ allows to conclude the proof. □

In the case where the initial matrix is Hermitian positive semidefinite, then the LU decomposition can be improved to a so-called "Cholesky decomposition" where $U = L^*$, the existence and uniqueness of L are simply deduced from the existence and uniqueness of a square root of A.

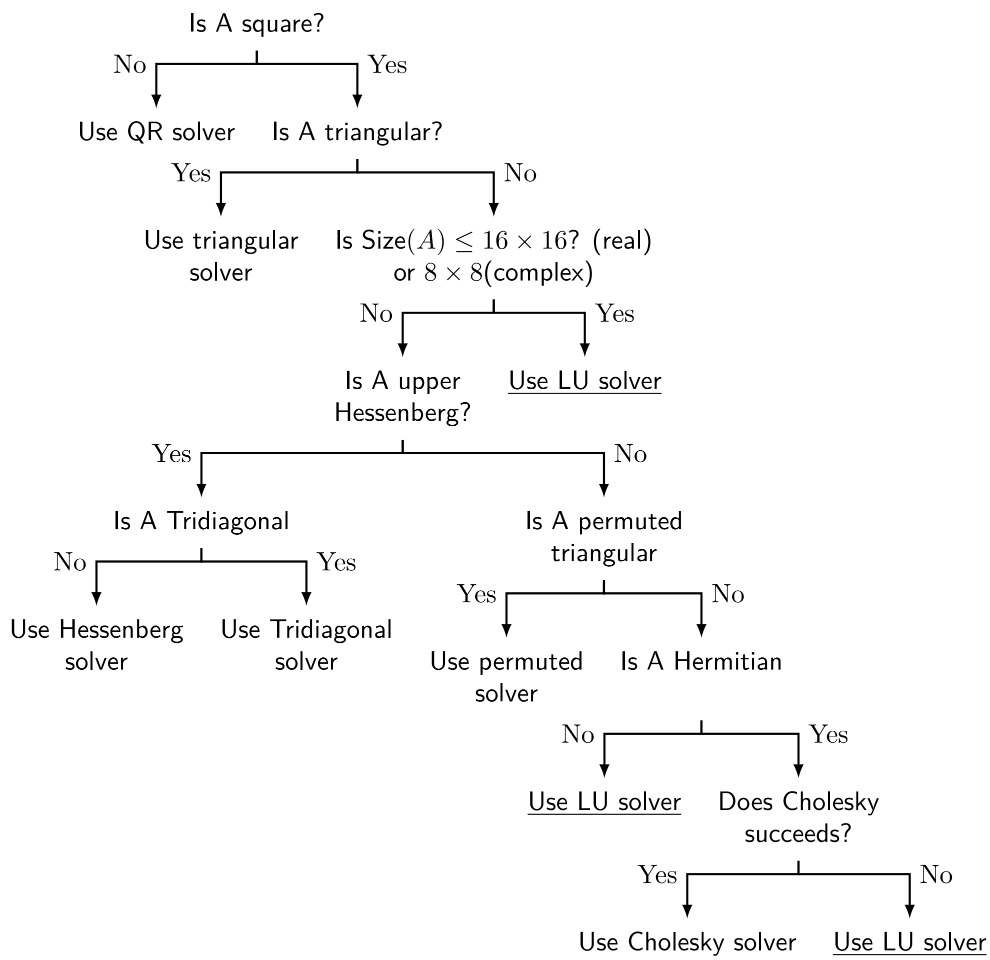


Figure 9.1: Chart taken from `https://www.mathworks.com//help//matlab/ref/mldivide.html` presenting the decision tree of the algorithm chosen to solve the system $Ax = B$ when A and B are full (i.e. not sparse).

Proposition 9.6 (Cholesky factorization)**.** *Let $A \in \mathcal{M}_n$ be Hermitian. Then A is positive semidefinite (respectively, positive definite) if and only if there is a lower triangular matrix $L \in \mathcal{M}_n$ with nonnegative (respectively, positive) diagonal entries such that $A = LL^*$. If A is positive definite, L is unique. If A is real, L may be taken to be real.*

Proof. Let $A^{1/2} = QR$ be a QR factorization and let $L = R^*$. Then $A = A^{1/2}A^{1/2} = R^*Q^*QR = R^*R = LL^*$.

In the case where A is PD, if there exists M lower triangular such that $A = MM^*$ then $M^{-1}L = M^{-*}L^{-*}$. Being both upper and lower triangular, $M^{-1}L$ and $M^{-*}L^{-*}$ are both equal to a diagonal matrix D. Knowing that all the term on the diagonal are positive real number, the identity writes more simply $D = M^{-1}L = ML^{-1}$ which implies $L = MD = DLD$, which can only be possible if $D = I_n$ since the diagonal terms of L are strictly positive. Finally, one has the identity $M = L$ which proves the uniqueness $\square$

The LU decomposition, together with the QR decomposition can be more or less interesting depending on the matrix considered. We display on Figure 9.1 the choices made by Matlab to find the solution x to the equation $Ax = B$. Similar chart exists to compute the eigen values of a matrix (as for the QR method and in a similar way, the LU decomposition can be used to compute the eigenvalue decomposition of a matrix).

Lecture 10

Kronecker product and Tensors

1 General results on Kronecker product

To give some motivation for the introduction of the Kronecker product, let us give us as objective to solve the matrix equation:

$$A_1XB_1 + A_2XB_2 = C$$

where the matrices A_1 A_2, B_1, B_2, C are given and X is unknown. In the description of the solutions of such equations, the Kronecker product, another product of matrices, is useful. In this chapter we develop the most important properties of this product and we study its application in the context of linear matrix equations. Note that the Kronecker product could be seen as a tensor product represented in a particular basis.

Definition 10.1 (Kronecker Product). *Given $A = [A_{ij}] \in \mathcal{M}_m$ and $B \in \mathcal{M}_n$, the Kronecker product of A and B is defined as:*

$$A \otimes B := [A_{ij}B] = \begin{bmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mm}B \end{bmatrix},$$

and is called the Kronecker product of A and B.

Note that this product is non commutative.

Lemma 10.1. *Given $A \in \mathcal{M}_n$, $B \in \mathcal{M}_m$, $C \in \mathcal{M}_p$, and $\mu \in \mathbb{C}$ the following computational rules hold:*

1. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.
2. $(\mu A) \otimes B = A \otimes (\mu B)$.
3. $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$, *whenever $A + B$ is defined.*
4. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$, *whenever $B + C$ is defined.*
5. $(A \otimes B)^T = A^T \otimes B^T$, *and therefore the Kronecker product of two symmetric matrices is symmetric.*

Note in particular that unike classical product the order of the matrices in the Kronecker product is not inverted through transposition.

Lemma 10.2 (Multiplication of Kronecker Products). *For $A, C \in \mathcal{M}_m$ and $B, D \in \mathcal{M}_n$ we have*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Hence, in particular:

The two first sections of this lecture is a close copy of the book of Jörg Liesen and Volker Mehrmann: Linear Algebra

1. $A \otimes B = (A \otimes I_n)(I_m \otimes B) = (I_m \otimes B)(A \otimes I_n)$,

2. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, *if* A *and* B *are invertible.*

Proof. Since $A \otimes B = [A_{ij}B]$ and $C \otimes D = [C_{ij}D]$, the block matrix $[F_{ij}] = (A \otimes B)(C \otimes D)$ is given by

$$F_{ij} = \sum_{k=1}^{m} (A_{ik}B)(C_{kj}D) = \sum_{k=1}^{m} A_{ik}C_{kj}BD = \left(\sum_{k=1}^{m} A_{ik}C_{kj}\right) BD.$$

For the block matrix $[G_{ij}] = (AC) \otimes (BD)$ with $G_{ij} \in \mathcal{M}_n$, we obtain

$$G_{ij} = H_{ij}BD, \text{ where } H_{ij} = \sum_{k=1}^{m} A_{ik}C_{kj},$$

which shows $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Items *1.* and *2.* easily follow from this equation. $\square$

Corollary 10.3. *Given* $A, P \in \mathcal{M}_m$ *and* $B, Q \in \mathcal{M}_n$ *where* P *and* Q *are invertible:*

$$(P \otimes Q)^{-1}(A \otimes B)(P \otimes Q) = (P^{-1}AP) \otimes (Q^{-1}BQ)$$

Lemma 10.4 (Non-Commutativity)**.** *In general, the Kronecker product is non-commutative, but for* $A \in \mathcal{M}_m$ *and* $B \in \mathcal{M}_n$ *there exists a permutation matrix* $P \in \mathcal{M}_{mn,mn}$ *such that*

$$A \otimes B = P^T(B \otimes A)P.$$

Proof. Given an integer $p \in \mathbb{N}$, we note $p_r^{(n)}$ the rest in the euclidean division of p with n and $p_q^{(n)}$ the quotient ($p = np_q^{(n)} + p_r^{(n)}$). To simplify the proof, we will now start the indexing of columns and rows of matrices from 0 ($A = (A_{i,j})_{i,j \in \{0,\dots,m-1\}}$ and $B = (B_{i,j})_{i,j \in \{0,\dots,n-1\}}$). Given $i, j \in \{0, \dots, nm\}$:

$$[A \otimes B]_{i,j} = A_{i_q^{(n)}, j_q^{(n)}} B_{i_r^{(n)}, j_r^{(n)}} \qquad \text{and} \qquad [B \otimes A]_{i,j} = A_{i_r^{(m)}, j_r^{(m)}} B_{i_q^{(m)}, j_q^{(m)}}, \tag{10.1}$$

Naturally, we introduce the permutation $\pi \in \mathfrak{S}_{nm}$ such that $\forall p \in [nm]$

$$\pi(p) = np_r^{(m)} + p_q^{(m)}.$$

The inequality $p \leq nm$ implies $p_q^{(m)} \leq n$, the uniqueness of the euclidean division rest and quotient then allows us to deduce that:

$$p_r^{(m)} = \pi(p)_q^{(n)} \qquad \text{and} \qquad p_q^{(m)} = \pi(p)_r^{(n)}.$$

Therefore, (10.1) allows us to set that:

$$[A \otimes B]_{i,j} = [B \otimes A]_{\pi(i),\pi(j)} = \left[P^T(B \otimes A)P\right]_{i,j}$$

where $P \in \mathcal{M}_{nm}$ is the permutation matrix[1] defined as $P = (\delta_{i,\pi(j)})_{i,j \in [nm]}$. $\square$

Theorem 10.5 (Properties of the Kronecker Product)**.** *For* $A \in \mathcal{M}_m$ *and* $B \in \mathcal{M}_n$ *the following rules hold:*

1. $\det(A \otimes B) = (\det A)^n (\det B)^m = \det(B \otimes A)$.

2. $Tr(A \otimes B) = Tr(A)\,Tr(B) = Tr(B \otimes A)$.

3. $Rk(A \otimes B) = Rk(A)Rk(B) = Rk(B \otimes A)$.

[1] Classically, $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$

Proof. 1. From *1.* in Lemma 10.2 and the multiplication theorem for determinants we get

$$\det(A \otimes B) = \det((A \otimes I_n)(I_m \otimes B)) = \det(A \otimes I_n)\det(I_m \otimes B).$$

It is straight forward ot see from the block diagonal matrix determinant formula that $\det(I_m \otimes B) = \det(B)^m$. By Lemma 10.4, there exists a permutation matrix P with $A \otimes I_n = P(I_n \otimes A)P^T$. This implies that

$$\det(A \otimes I_n) = \det(P(I_n \otimes A)P^T) = \det(I_n \otimes A) = (\det A)^n.$$

It then follows that $\det(A \otimes B) = (\det A)^n(\det B)^m$, and therefore also $\det(A \otimes B) = \det(B \otimes A)$.

2. From $A \otimes B = [A_{ij}B]$ we obtain

$$\mathrm{Tr}(A \otimes B) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ii}B_{jj} = \left(\sum_{i=1}^{m} A_{ii}\right)\left(\sum_{j=1}^{n} B_{jj}\right) = \mathrm{Tr}(A)\mathrm{Tr}(B)$$
$$= \mathrm{Tr}(B)\mathrm{Tr}(A) = \mathrm{Tr}(B \otimes A).$$

3. We know from Schur Theorem that there exists some matrices $P, T \in \mathcal{M}_m$ and $Q, U \in \mathcal{M}_n$ such that P, Q are both invertible, T, U both upper triangular and:

$$A = P^{-1}TP \qquad \text{and} \qquad B = Q^{-1}UQ.$$

The rank of A and B are respectively the number of zeros on the diagonal of T and U. Corollary 10.3 allows us to express:

$$\mathrm{Rk}(A \otimes B) = \mathrm{Rk}\left((P \otimes Q)^{-1}(A \otimes B)(P \otimes Q)\right) = \mathrm{Rk}(T \otimes U) = \mathrm{Rk}(A)\mathrm{Rk}(B),$$

thanks to a simple account of the number of zeros on the diagonal.

□

2 Resolution of linear matrix equations

For a matrix $A = [a_1, \dots, a_n] \in \mathcal{M}_{m,n}$ with columns $a_j \in \mathbb{C}^m$, $j = 1, \dots, n$, we define

$$\mathrm{Vec}(A) := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{C}^{mn}.$$

The application of "vec" turns the matrix A into a "column vector" and thus "vectorizes" A.

Theorem 10.6 (Vectorization and Kronecker Product)**.** *For $A \in \mathcal{M}_m$, $B \in \mathcal{M}_n$, and $C \in \mathcal{M}_{m,n}$ we have*

$$Vec(ACB) = (B^T \otimes A)\, Vec(C).$$

Hence, in particular,

1. *$Vec(AC) = (I_n \otimes A)\, Vec(C)$ and $Vec(CB) = (B^T \otimes I_m)\, Vec(C)$,*
2. *$Vec(AC + CB) = ((I_n \otimes A) + (B^T \otimes I_m))\, Vec(C)$.*

Proof. For $j = 1, \dots, n$, the jth column of ACB is given by

$$(ACB)e_j = (AC)(Be_j) = \sum_{k=1}^{n} B_{kj}(AC)e_k = \sum_{k=1}^{n}(B_{kj}A)(Ce_k)$$
$$= \left[B_{1j}A, B_{2j}A, \dots, B_{nj}A\right]\mathrm{Vec}(C),$$

which implies that $\mathrm{Vec}(ACB) = (B^T \otimes A)\mathrm{Vec}(C)$. With $B = I_n$ respectively $A = I_m$ we obtain *1.*, while *1.* and the linearity of vec yield *2.*. □

In order to study the relationship between the eigenvalues of the matrices A, B and those of the Kronecker product $A \otimes B$, we use bivariate polynomials, i.e., polynomials in two variables. If

$$p(t_1, t_2) = \sum_{i,j=0}^{l} \alpha_{ij} t_1^i t_2^j \in \mathbb{C}[t_1, t_2]$$

is such a polynomial, then for $A \in \mathcal{M}_m$ and $B \in \mathcal{M}_n$ we define the matrix

$$p(A, B) := \sum_{i,j=0}^{l} \alpha_{ij} A^i \otimes B^j.$$

Here we have to be careful with the order of the factors, since in general $A^i \otimes B^j \neq B^j \otimes A^i$.

Theorem 10.7 (Stephanos)**.** *Let $A \in \mathcal{M}_m$ and $B \in \mathcal{M}_n$ be two matrices that have Jordan canonical forms and the eigenvalues $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $\mu_1, \ldots, \mu_n \in \mathbb{C}$, respectively. The following assertions hold:*

1. *The eigenvalues of $p(A, B)$ are $p(\lambda_k, \mu_\ell)$ for $k = 1, \ldots, m$ and $\ell = 1, \ldots, n$.*
2. *The eigenvalues of $A \otimes B$ are $\lambda_k \cdot \mu_\ell$ for $k = 1, \ldots, m$ and $\ell = 1, \ldots, n$.*
3. *The eigenvalues of $A \otimes I_n + I_m \otimes B$ are $\lambda_k + \mu_\ell$ for $k = 1, \ldots, m$ and $\ell = 1, \ldots, n$.*

Proof. Let $S \in \mathcal{M}_m(\mathbb{C})$ and $T \in \mathcal{M}_n(\mathbb{C})$ be invertible such that $S^{-1}AS = J_A$ and $T^{-1}BT = J_B$ are in Jordan canonical form. The matrices J_A and J_B are upper triangular. Thus, for all i, j the matrices $J_A^i \otimes J_B^j$ and $J_A^i \otimes J_B^j$ are upper triangular. The eigenvalues of J_A and J_B are $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$, respectively. Thus, $p(\lambda_k, \mu_\ell)_{k \in [m], \ell \in [n]}$, are the diagonal entries of the matrix $p(J_A, J_B)$. Using Corollary 10.3 we obtain

$$\begin{aligned} p(A, B) &= \sum_{i,j=0}^{l} \alpha_{ij} (SJ_AS^{-1})^i \otimes (TJ_BT^{-1})^j = \sum_{i,j=0}^{l} \alpha_{ij} (SJ_A^iS^{-1}) \otimes (TJ_B^jT^{-1}) \\ &= \sum_{i,j=0}^{l} \alpha_{ij} (S \otimes T)(J_A^i \otimes J_B^j)(S \otimes T)^{-1} \quad = \quad (S \otimes T) p(J_A, J_B)(S \otimes T)^{-1} \end{aligned}$$

which implies *1.* The assertions *2.* and *3.* follow from *1.* with $p(t_1, t_2) = t_1 t_2$ and $p(t_1, t_2) = t_1 + t_2$, respectively. ☐

Lemma 10.8 (Matrix Exponential of a Kronecker Product)**.** *For $A \in \mathcal{M}_m$, $B \in \mathcal{M}_n$, and $C := (A \otimes I_n) + (I_m \otimes B)$ we have*

$$\exp(C) = \exp(A) \otimes \exp(B).$$

Proof. From Lemma 10.2, Item *1.*, we know that the matrices $A \otimes I_n$ and $I_m \otimes B$ commute. With classical operation on exponentials:

$$\begin{aligned} \exp(C) &= \exp(A \otimes I_n + I_m \otimes B) \\ &= \exp(A \otimes I_n) \exp(I_m \otimes B) \\ &= \left(\sum_{j=0}^{\infty} \frac{1}{j!} (A \otimes I_n)^j \right) \left(\sum_{i=0}^{\infty} \frac{1}{i!} (I_m \otimes B)^i \right) \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{1}{j! i!} (A \otimes I_n)^j (I_m \otimes B)^i \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{1}{j! i!} (A^j \otimes B^i) \\ &= \exp(A) \otimes \exp(B), \end{aligned}$$

where we have used the properties of the matrix exponential series. ☐

The following result on the matrix exponential function of a Kronecker product is helpful in applications that involve systems of linear differential equations. For given matrices $A_j \in \mathcal{M}_m$, $B_j \in \mathcal{M}_n$, $j = 1, \dots, q$, and $C \in \mathcal{M}_{m,n}$ an equation of the form

$$A_1 X B_1 + A_2 X B_2 + \ldots + A_q X B_q = C \tag{10.2}$$

is called a linear matrix equation for the unknown matrix $X \in \mathcal{M}_{m,n}$.

Theorem 10.9. *A matrix $\hat{X} \in \mathcal{M}_{m,n}$ solves (10.2) if and only if $Vec(\hat{X}) \in \mathcal{M}_{mn,1}$ solves the linear system of equations*

$$G\,Vec(\hat{X}) = Vec(C), \text{ where } G := \sum_{j=1}^{q} B_j^T \otimes A_j.$$

Proof. Simple application of Theorem 10.6. $\square$

We now consider two special cases of (10.2).

Theorem 10.10 (Sylvester Equation)**.** *For $A \in \mathcal{M}_m$, $B \in \mathcal{M}_n$, and $C \in \mathcal{M}_{m,n}$, the Sylvester equation*

$$AX + XB = C$$

has a unique solution if and only if A and $-B$ have no common eigenvalue.

Proof. Analogous to the representation in Theorem 10.9, we can write the Sylvester equation as

$$(I_n \otimes A + B^T \otimes I_m)\mathrm{Vec}(X) = \mathrm{Vec}(C),$$

then Theorem 10.7 allows us to set that the set of eigenvalues of $I_n \otimes A + B^T \otimes I_m$ is exactly $\{\lambda + \mu, \lambda \in \mathrm{Sp}(A), \mu \in \mathrm{Sp}(B)\}$ which does not contains 0 if and only if A and $-B$ do not have common eigenvalue. $\square$

Corollary 10.11. *For $A, C \in \mathcal{M}_n$ the Lyapunov equation*

$$AX + XA^* = C$$

has a unique solution $\hat{X} \in \mathbb{C}^{n,n}$ if the eigenvalues of A have negative real parts. If, furthermore, C is Hermitian, then also $\hat{X}$ is Hermitian.

Proof. Since by assumption A and $-A^*$ have no common eigenvalues, the unique solvability follows from Theorem 10.10. Note besides that if C is Hermitian, then:

$$A\hat{X}^* + \hat{X}^* A^* = C^* = C,$$

in other words, $\hat{X}^*$ is also solution, which implies, by uniqueness of the solution that $\hat{X}^* = \hat{X}$ ($\hat{X}$ is Hermitian). $\square$

Equations provided in Theorem 10.10 and Corollary 10.11 are quite important in the field of control theory (that deals with the control of dynamical systems in engineered processes and machines), therefore some powerful method are required to compute the solution. A standard solution is to employ the Bartels–Stewart algorithm that relies on the triangulation provided by the Schur decomposition. We simply describe below the procedure to compute solution to:

$$AX + XA^* = C.$$

Following the result of Theorem 10.10, we assume A and $-A^*$ do not have common eigenvalues so that the equation admits a unique solution.

1. Compute the Schur decomposition $R = U^* A U$ where the matrix R is upper triangular and U is Hermitian.

2. Set $F = U^T C V$, one then has to solve the simplified system $RY + YR^* = F$
3. Consider the block decomposition $F = \left(\begin{smallmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{smallmatrix}\right)$, $R = \left(\begin{smallmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{smallmatrix}\right)$, where $F_{22}, Y_{22}, R_{22} \in \mathbb{C}$ (then $F_{11}, Y_{11}, R_{11} \in \mathcal{M}_{n-1}$), one then has to solve the system of equations:

$$\begin{aligned} F_{11} &= R_{11}Y_{11} + Y_{11}R_{11}^* + R_{12}Y_{21} + Y_{12}R_{12}^* \\ F_{12} &= R_{11}Y_{12} + R_{12}Y_{22} + Y_{12}R_{22}^* \\ F_{21} &= R_{22}Y_{21} + Y_{21}R_{11}^* + Y_{22}R_{12}^* \\ F_{22} &= R_{22}Y_{22} + Y_{22}R_{22}^*. \end{aligned}$$

4. The last equation imposes $Y_{22} = \frac{F_{22}}{R_{22}+R_{22}^*}$ ($R_{22} + R_{22}^* \neq 0$ since R_{22} is an eigenvalue of A that should therefore satisfy $-R_{22}^* \neq R_{22}$), then Y_{21} and Y_{12} are solved with the equations:

$$Y_{21}(R_{22}I_{n-1} - R_{11}^*) = F_{21} - Y_{22}R_{12}^* \qquad \text{and} \qquad (R_{11} + R_{22}^* I_{n-1})Y_{12} = F_{12} - Y_{22}R_{21},$$

which is easy to solve by iterative substitution since $R_{11} + R_{22}^* I_{n-1}$ is triangular. Note that $R_{22}I_{n-1} + R_{11}^*$ is invertible because by hypothesis on A, R_{22} is not an eigenvalue of $-A^*$, thus of R_{11}^*.

5. Employ the previous procedure (starting with a decomposition in $n-1, 1$ block matrices) to the equation $R_{11}Y_{11} + Y_{11}R_{11}^* = F_{11} - R_{12}Y_{21} - Y_{12}R_{12}^*$ and proceed until one gets a block decomposition $1, 1$.

3 Tensors

Tensor is a multi-way array. An N-way tensor $\mathcal{T} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ can express:

$$\mathcal{T} = (T_{i_1, i_2, \ldots, i_N})_{i_1 \in [I_1], \ldots, i_N \in [I_N]}.$$

(in particular matrices are 2-way tensors). We will focus on 3-way tensor.

Definition 10.2. *(Outer product) Given $a \in \mathbb{C}^I$ and $b \in \mathbb{C}^J$, we denote $a \bullet b = ab^T$. Given a supplementary $c \in \mathbb{C}^K$:*

$$a \bullet b \bullet c = (a_i b_j c_k)_{i \in I, j \in J, k \in K}.$$

One of the big issues in tensor theory is to represent a tensor $\mathcal{X}$ as a following sum:

$$\mathcal{X} = \sum_{r=1}^{R} (a_r \bullet b_r \bullet c_r), \tag{10.3}$$

where $R \in \mathbb{N}$. There exist multiple definitions of the rank, we provide below the most common one that relies on the above decomposition.

Definition 10.3 (Tensor rank)**.** *Given a tensor $T \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$, the minimal integer $R \in \mathbb{N}$ such that there exists RN vectors:*

$$a_1^{(1)}, \ldots, a_R^{(1)} \in \mathbb{C}^{I_1}, \qquad a_1^{(2)}, \ldots, a_R^{(2)} \in \mathbb{C}^{I_2}, \qquad \ldots, \qquad a_1^{(N)}, \ldots, a_R^{(N)} \in \mathbb{C}^{I_N},$$

such that $T = \sum_{r=1}^R a_r^{(1)} \bullet \cdots \bullet a_r^{(N)}$.

Then we say that T is a sum of R tensor of rank 1.

Proof of the existence and uniqueness of the rank. Given $i_1 \in [I_1], \ldots, i_N \in [I_N]$, let us introduce the tensor:

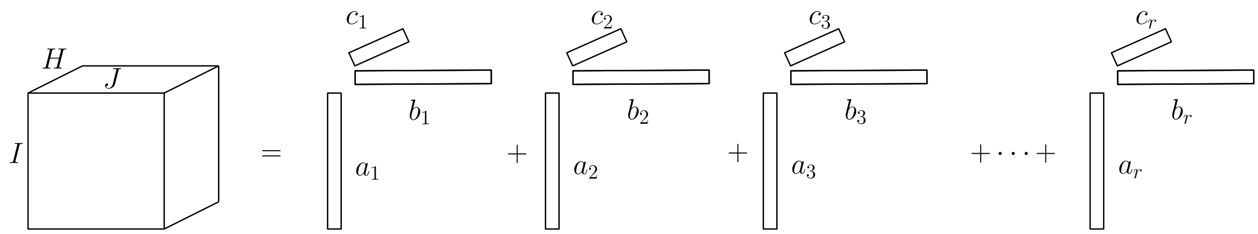
$$E_{i_1, \ldots, i_N} \equiv e_{i_1}^{I_1} \bullet \cdots \bullet e_{i_N}^{I_N} \in \mathbb{C}^{I_1 \times \cdots \times I_N},$$

where for all $\ell \in [N]$, $e_{i_\ell}^{I_\ell}$ is the i_ℓ vector of the canonical basis of $\mathbb{C}^{I_\ell}$. Note that $E_{i_1,\dots,i_N}$ is full of zeros with just an entry equal to one at the index $(i_1,\dots,i_N)$. The tensors $(E_{i_1,\dots,i_N})_{i_1\in[I_1],\dots,i_N\in[I_N]}$ form a basis of $\mathbb{C}^{I_1\times\cdots\times I_N}$ and for any $T \in \mathbb{C}^{I_1\times\cdots\times I_N}$, we have:

$$T = \sum_{i_1\in[I_1],\dots,i_N\in[I_N]} T_{i_1,\dots,i_N} E_{i_1,\dots,i_N} = \sum_{i_1\in[I_1],\dots,i_N\in[I_N]} T_{i_1,\dots,i_N} e_{i_1}^{I_1} \bullet \cdots \bullet e_{i_N}^{I_N},$$

which implies that the rank is well defined as the minimum of a non empty set bounded below by 0. Note in passing that we have just shown that the rank of a tensor of $\mathbb{C}^{I_1\times\cdots\times I_N}$ is always lower that $I_1\cdots I_N$. $\square$

Decomposition (10.3) can be schematically represented this way:



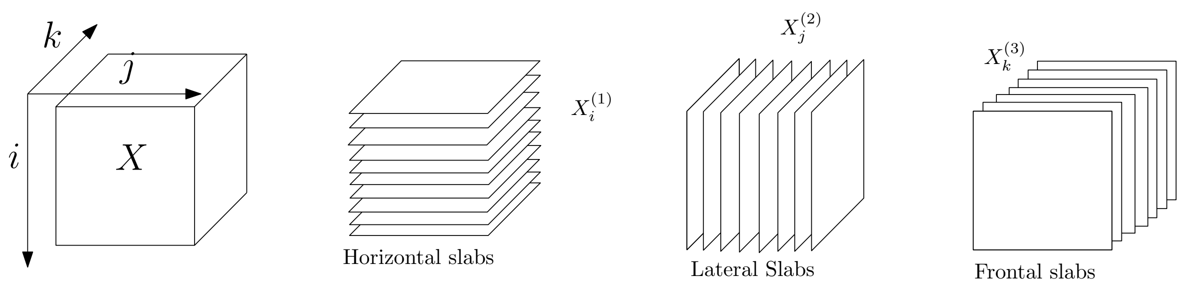
This decomposition is not unique and it is also known as "tensor rank decomposition", "Canonical Polyadic decomposition" (CPD) or "Parallel factor analysis"

The tensor decomposition problem formulates:

$$\text{Minimize:} \quad \left\| \mathcal{X} - \sum_{r=1}^{R} a_r \bullet b_r \bullet c_r \right\|_F^2, \tag{10.4}$$

where $[a_1, a_2, \dots, a_R] \in \mathbb{C}^{I\times R}$, $[b_1, b_2, \dots, b_R] \in \mathbb{C}^{J\times R}$, $[c_1, c_2, \dots, c_R] \in\in \mathbb{C}^{K\times R}$.

This problem can be rewritten with so-called "slabs" which can be defined through 3 directions (for tensors of degree 3) and are simply obtained by fixing one index of a given axis. We depict below the slabs of all directions:



given a decomposition $\mathcal{X} = \sum_{r=1}^{R}(a_r \bullet b_r \bullet c_r)$, note that:

$$X_i^{(1)} = \sum_{r=1}^{R} a_{ir}(b_r \bullet c_r) = \sum_{r=1}^{R} a_{ir} b_r c_r^T \qquad X_j^{(2)} = \sum_{r=1}^{R} b_{jr} a_r c_r^T \qquad X_k^{(3)} = \sum_{r=1}^{R} c_{kr} a_r b_r^T.$$

The problem can then rewrite:

$$\text{Minimize:} \quad \sum_{i=1}^{I} \left\| \mathcal{X}_i^{(1)} - \sum_{r=1}^{R} A_{r,i} b_r c_r^T \right\|_F^2,$$

where we employed the notation $A = (a_1,\dots,a_R) = (A_{i,r})_{i\in[I],r\in[R]}$.

Note with a vectorization that given $i \in [I]$:

$$\mathrm{Vec}(X_i^{(1)}) = \sum_{r=1}^{R} a_{ir} \mathrm{Vec}(b_r c_r^T) = \underbrace{(\mathrm{Vec}(b_1 c_1^T), \dots, \mathrm{Vec}(b_R c_R^T))}_{=(C\odot B)\in\mathbb{R}^{JK\times R}} \begin{pmatrix} a_{i,1} \\ \vdots \\ a_{i,R} \end{pmatrix} = (C \odot B)\tilde{a}_i,$$

where the notation "$C \odot B$" designates the Khatri-Rao product product. Note that for any vector $a \in \mathbb{R}^I$ and $b \in \mathbb{R}^J$, $\mathrm{Vec}(ab^T) = b \otimes a$ so the Khatri-Rao product of two matrices $A \in \mathbb{R}^{I\times R}$ and $B \in \mathbb{R}^{J\times R}$ can also be expressed followingly:

$$A \odot B = (a_1 \otimes b_1, \ldots, a_R \otimes b_R).$$

Note also that $\tilde{a}_i$ is actually the i^{th} column of A^T. Thus, if one introduces the matrix:

$$\mathcal{X}^{(1)} \equiv \left(\mathrm{Vec}(X_1^{(1)}), \ldots, \mathrm{Vec}(X_I^{(1)})\right),$$

one has the identity:

$$\mathcal{X}^{(1)} = (C \odot B)A^T$$

The same way, with similar notations, one can show that:

$$\mathcal{X}^{(2)} = (C \odot A)B^T \qquad \text{and} \qquad \mathcal{X}^{(3)} = (B \odot A)C^T.$$

The problem (10.4), is then generally solved by the so-called "Alternating Least Squares (ALS) Algorithm". The idea is to fix all factor matrices except for one in order to optimize for the non-fixed matrix with a classical least square algorithm and then repeat this step for every matrix repeatedly until some stopping criterion is satisfied. More precisely, for 3-way tensor case one needs to follow the following steps repeatedly until convergence:

$$\begin{aligned}
A &\leftarrow \arg\min_A \|\mathcal{X}^{(1)} - (C \odot B)A^T\| \\
B &\leftarrow \arg\min_B \|\mathcal{X}^{(2)} - (C \odot A)B^T\| \\
C &\leftarrow \arg\min_C \|\mathcal{X}^{(3)} - (B \odot A)C^T\|
\end{aligned}$$