Homework 2: Jordan decomposition and exponential of matrices

General instructions

- You must submit your results and solutions in a single PDF file. We recommend using LaTeX with a basic template, which you can reuse for each homework assignment.
- This assignment is due on 07/03/2024 by 23:00:00. Submissions will not be accepted after this deadline.
- If you have any questions, you can contact TA Zhang Wenrang via email at 223040237@link.cuhk.edu.cn, or during office hours on Wednesday from 14:00 to 15:00 in room ZX, 4F-72.

Problem 1: Jordan decomposition

The goal of this problem is to write the Jordan decomposition of the matrix:

$$B = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

1. Compute the characteristic polynomial of B and deduce the possible Jordan decompositions of B
   Correction: *Its characteristic polynomial (developed along the first column) is:*

$$\begin{aligned} \chi_B &= (1-X)\cdot[-X(1-X)-1] + 1\cdot(1-X) \\ &= (1-X)\cdot(X^2-X) \\ &= \boxed{-X(X-1)^2} \end{aligned}$$

   It follows that 0 is an eigenvalue of algebraic multiplicity 1, and 1 is an eigenvalue of algebraic multiplicity 2; notably B is triangularizable and possesses a Jordan reduction over $\mathbb{R}$. Given the direct decomposition into characteristic spaces, there are essentially two possible forms for the reduction:

$$\boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \text{ (diagonalizable)} \qquad \boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \text{ (non-diagonalizable)}$$

   Remember that the characteristic polynomial does not allow us to distinguish them. $\square$

2. Express the eigenspaces of B and provide the algebraic and geometric multiplicities of the eigenvalues of B.

Correction: *Firstly,* $E_0(B) = \ker B = \ker \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} = Vect(v_1)$, *with* $\boxed{v_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}$*.Now,* $E_1(B) = \ker(B - I_3) = \ker \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} = Vect(v_2)$, *with* $\boxed{v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$. *Notably, this teaches us about its minimal polynomial: equal to the characteristic polynomial, and its Jordan form:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The algebraic multiplicity of 0 *and* 1 *are respectively* $\boxed{1 \text{ and } 2}$ *and their geometric multiplicity is respectively* $\boxed{1 \text{ and } 1}$. □

3. Find two vectors $v_2, w_3 \in \ker(B - I_3)^2$ such that $Bv_2 = v_2$ and$Bw_3 = v_2 + w_3$. Deduce a change of basis matrix P that allows to find the Jordan decomposition of B (no need to invert P).

Correction: *We can directly take* $v_2 = \beta\left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right)$ *for any* $\beta \neq 0$ *from the previous part. By solving* $Bw_3 = v_2 + w_3$ *we obtain* $w3 = \beta\left(\begin{smallmatrix} \gamma \\ 1 \\ -1-\gamma \end{smallmatrix}\right)$ *with the same* β *of* v_2 *and any* $\gamma \in \mathbb{R}$.
As for the matrix P,

- *the column* $\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ *in* J *should correspond to the 0-eigenbasis that you found in the previous part.*
- *the columns* $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ *in* J *should correspond to* v_2 *and* w_3.

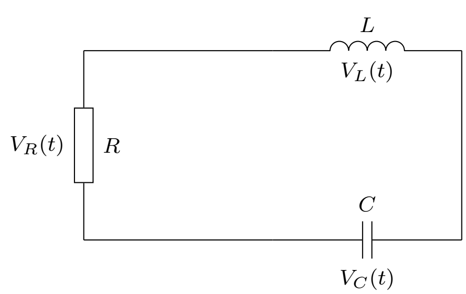
For example:

$$P^{-1}BP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & -1 \\ -1 & 1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -0.5 & 1 & 0 \\ 0.5 & 0 & 1 \\ 0.5 & -1 & -1 \end{pmatrix} \cdots$$

$$P^{-1}BP = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \text{ or } \cdots$$

□

Problem 2: Dynamic of an electrical network



Noting I the intensity in the circuit, one has the following relations that model the dynamic behavior of the circuit:

$$L\frac{dI}{dt} = V_L,$$
$$C\frac{dV_C}{dt} = -I,$$
$$RI = V_R,$$
$$V_L + V_C + V_R = 0.$$

1. Express matricially the differential system of equation of $y = \begin{pmatrix} I \\ V_C \end{pmatrix}$. [Hint] Establish an equation between the differential of y and itself.

   Correction: *Noting* $A = \begin{pmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{pmatrix}$, $y(t) = \begin{pmatrix} I(t) \\ V_C(t) \end{pmatrix}$ *and* $y_0 = \begin{pmatrix} I_0 \\ 0 \end{pmatrix}$, *the differential equation system becomes* $\frac{dy}{dt} = Ay$. □

2. Solve the system with initial condition $y(0) = \begin{pmatrix} I_0 \\ 0 \end{pmatrix}$. We want a result without exponential of matrices.

   Correction: *The solution of the system*

$$\begin{cases} \dfrac{dy}{dt} = Ay \\ y(0) = y_0 \end{cases}$$

is $y(t) = e^{tA}y_0$, *we are then just left to compute* e^{tA}. *To find the Jordan decomposition of* A, *let us first compute:*

$$\chi_A = \left(X + \frac{R}{L}\right)X + \frac{1}{CL} = X^2 + \frac{R}{L}X + \frac{1}{CL}.$$

There exists two complex roots α_1, α_2 *easy to compute such that* $\chi_A = (X - \alpha_1)(X - \alpha_2)$. *Let us first assume that* $\alpha_1 \neq \alpha_2$. *Then* A *is diagonalizable and we then look for* $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ *such that* $Au = \alpha_1 u$. *That leads us to solving:*

$$\begin{cases} -\dfrac{R}{L}u_1 + \dfrac{u_2}{L} & = \alpha_1 u_1 \\ -\dfrac{u_1}{C} & = \alpha_1 u_2 \end{cases} \tag{1}$$

Note that:

$$\frac{R}{L} = -\alpha_1 - \alpha_2 \qquad \text{and} \qquad \frac{1}{LC} = \alpha_1\alpha_2. \tag{2}$$

It is then clear that the couple of solutions $u_1 = 1$ *and* $u_2 = -L\alpha_2$ *satisfies the system* (1). *The same way, one sees that* $w = \begin{pmatrix} 1 \\ -L\alpha_1 \end{pmatrix}$ *is solution to* $Aw = \alpha_2 w$. *Then the matrix* $P \equiv (u, w) \in \mathcal{M}_2$ *allow us to diagonalize:*

$$P^{-1}AP = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$

As a consequence, the solution to our electric dynamic system is:

$$y(t) = P^{-1}\begin{pmatrix} e^{\alpha_1 t} & 0 \\ 0 & e^{\alpha_2 t} \end{pmatrix} P \begin{pmatrix} I_0 \\ 0 \end{pmatrix}$$

When $\alpha_1 = \alpha_2 \equiv \alpha$, we know that A is not diagonalizable because otherwise one would have $A = \alpha I_2$. In order to find a basis in which the matrix A takes the canonical Jordan form $\left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right)$, we look for a vector $x = \left(\begin{smallmatrix} 1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}$ such that $Ax = w + \alpha x$ with $w = \left(\begin{smallmatrix} 1 \\ -L\alpha \end{smallmatrix}\right)$. That leads to the equation:

$$\begin{cases} -\frac{R}{L} + \frac{x_2}{L} = 1 + \alpha \\ \qquad -\frac{1}{C} = -L\alpha + \alpha x_2, \end{cases} \tag{3}$$

thanks to the identities $\alpha = -\frac{R}{2L}$ and $\alpha^2 = \frac{1}{CL}$, it is not hard to see that $x_2 \equiv L(1-\alpha)$ is solution to (3). Finally writing $P \equiv (w, x)$, one has the decomposition:

$$P^{-1}AP = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix},$$

which allows us to conclude that:

$$\begin{aligned} y(t) &= P^{-1} \exp\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) P \begin{pmatrix} I_0 \\ 0 \end{pmatrix} = P^{-1} \exp\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P \begin{pmatrix} I_0 \\ 0 \end{pmatrix} \\ &= P^{-1} \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha t} \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} P \begin{pmatrix} I_0 \\ 0 \end{pmatrix} = P^{-1} \begin{pmatrix} e^{\alpha t} & te^{\alpha t} \\ 0 & e^{\alpha t} \end{pmatrix} P \begin{pmatrix} I_0 \\ 0 \end{pmatrix} \end{aligned}$$

□

Exercises

1. Given $A \in \mathcal{M}_n$, $B \in \mathcal{M}_m$ and $C \in \mathcal{M}_{n,m}$ such that $\text{Rank}(C) = m$ and $AC = CB$, prove that all the eigen values of B are eigen values of A.

   Correction: *For any eigen-pair (v, λ) of B, we have $Bv = \lambda v$, thus $ACv = CBv = \lambda Cv$, which means $\exists y = Cv$ such that $Ay = \lambda y$ therefore λ is an eigenvalue of A.* □

2. Express the sufficient and necessary conditions on the coefficients $a, b, c, d \in \mathbb{R}$ such that the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is triangularizable with a unitary matrix. Same question for A to be diagonalizable (with any matrix).

   Correction: *The characteristic polynomial of A is $\chi_A = (X-a)(X-d) - bc = X^2 - (a+d)X + ad - bc$. For A to be triangularizable with an unitary matrix, the necessary and sufficient condition is that A has real roots which means $(a+d)^2 + 4ad + 4bc \geq 0$.*
   A sufficient condition for A to be diagonalizable is A having distinct roots ($(a+d)^2 - 4ad + 4bc > 0$). If $(a+d)^2 - 4ad + 4bc = 0$, the matrix can have just one eigenvalue equal to $\lambda \equiv \frac{a+d}{2}$ then note that for any invertible matrix $P \in \mathcal{M}_2$:

   $$P^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} P = P^{-1}(\lambda I_2)P = \lambda I_2,$$

   thus the only possibility for A to be diagonalizable in that case is that $b = c = 0$. To sum up with a sufficent and necessary condition for A to be diagonalizable is:

   $$b = c = 0, a = d \qquad \textit{or} \qquad (a+d)^2 - 4ad + 4bc > 0.$$

   □

3. Find all the canonical forms that can have a matrix A having the characteristic polynomial:

$$\chi_A = (X-2)^3(X+1)^2$$

Correction: *The different Jordan decomposition possible for A are:*

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix},$$

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}$$

□

4. What is a necessary condition of two matrices A, B such that $e^{A+B} \neq e^A e^B$ (No need for proof)? Give one example pair of such matrices in $A, B \in \mathcal{M}_2$.

5. Prove that for any $A \in \mathcal{M}_n(\mathbb{C})$, $\det(e^A) = e^{\mathrm{Tr}(A)}$.

Correction:
Proof 1:

(a) *Any matrix A in $\mathcal{M}_n(\mathbb{C})$ can have some Jordan decomposition $A = SJS^{-1}$. Therefore we have $e^A = Se^J S^{-1}$ and* $\boxed{\det(e^A) = \det(e^J)}$.

(b) *Since J is upper triangular, we know J^k is upper triangular for all $k \in \mathbb{N}_{++}$, therefore e^J is upper triangular. We have* $\boxed{\det(e^J) = \Pi_i [e^J]_{ii}}$ *where $[e^J]_{ii}$ is the ith diagonal element of e^J.*

(c) *For any 2 upper triangular matrix P and Q, we demonstrate $[PQ]_{ii} = P_{ii}Q_{ii}$:*

$$[PQ]_{ii} = \sum_j P_{ij}Q_{ji} \overset{P \text{ is upper tri}}{=\!=\!=} \sum_{j \geq i} P_{ij}Q_{ji} \overset{Q \text{ is upper tri}}{=\!=\!=} \sum_{j=i} P_{ij}Q_{ji} = P_{ii}Q_{ii}$$

Combined with the additive construction of e^J we have $\boxed{[e^J]_{ii} = \left[\sum_k \frac{J^k}{k!}\right]_{ii} = \sum_k \frac{J^k_{ii}}{k!} = e^{J_{ii}}}$

(d) *Therefore $\det(e^A) = \det(e^J) = \Pi_i [e^J]_{ii} = \Pi_i e^{J_{ii}} = e^{Tr(J)}$. We finally have* $\boxed{\det(e^A) = e^{Tr(A)}}$ *since similar matrices have same trace.*

Proof 2:
Any matrix A in $\mathcal{M}_n(\mathbb{C})$ can have some Jordan decomposition $A = SJS^{-1}$. Therefore we have $e^A = Se^J S^{-1}$ and $\boxed{\det(e^A) = \det(e^J)}$.

Denote $J = \begin{pmatrix} J_{d_1}(\lambda_1) & & (0) \\ & \ddots & \\ (0) & & J_{d_k}(\lambda_k) \end{pmatrix}$. *Let us construct* $J_i = \begin{pmatrix} (0) & & & & \\ & \ddots & & & \\ & & J_{d_i}(\lambda_i) & & \\ & & & \ddots & \\ & & & & (0) \end{pmatrix}$ *for all*

$i \in [k]$ and we have $J = \sum_{i=1}^{k} J_i$. We can easily prove that J_i and J_l commute for all $i, l \in [k]$ thus $\boxed{\det(e^J) = \det(e^{\sum_{i=1}^{k} J_i}) = \det(\Pi_{i=1}^{k} e^{J_i}) = \Pi_{i=1}^{k} \det(e^{J_i})}$. Let us inspect $\det(e^{J_i})$: $e^{J_i} = \sum_{k=0}^{\infty} \frac{J_i^k}{k!}$; since J_i is upper triangular, e^{J_i} is also upper triangular and its determinant is equal to the product of its diagonal entries. As a consequence, the n-th diagonal value of e^{J_i} is $\sum_{k=0}^{\infty} \frac{[J_i]_{nn}}{k!} = e^{[J_i]_{nn}}$ where $[J_i]_{nn}$ is the n-th diagonal value of J_i. therefore $\boxed{\det(e^{J_i}) = \Pi_n e^{[J_i]_{nn}} = e^{Tr(J_i)}}$. Finally:

$$\det(e^A) = \Pi_{i=1}^{k} e^{Tr(J_i)} = e^{\sum_{i=1}^{k} Tr(J_i)} = e^{Tr(J)} = \boxed{e^{Tr(A)}}$$

since similar matrices have same trace. $\square$