Final Exam

May, 13^{th} 2024

- Time Limit: 10:30 am - 12:30 pm.
- **No** books, course notes nor electronic devices are allowed.
- The problems are on the other side of the paper.
- Upon finished, the examination paper has to be submitted together with your answer book.

Course check (40%)

1. Define what does it mean that two norms are equivalent. What happens in finite dimensional vector spaces like $\mathcal{M}_n$?

2. Give the definition of matrix norm and spectral radius, give their relation and prove it.

3. Given a matrix A with all entries strictly positive, what can be deduced on the eigenvalues of biggest modulus and on their associated eigenspace? (Put together the 3 Perron Frobenius results).

4. Given $A, B \in \mathcal{H}_n$, define the notation $A \succeq B$ and prove:

   - Given any invertible matrix[1] $P \in \mathcal{M}_{p,n}$: $P^*AP \succeq B \implies A \succeq P^{-*}BP^{-1}$.
   - $A \succeq B \succ 0 \iff 0 \prec A^{-1} \preceq B^{-1}$.

5. Provide the definition of the tensor rank.

Problem 1 (25%): Normal matrices.

In this problem we will work with so called "normal matrices" that are matrices $A \in \mathcal{M}_n(\mathbb{C})$ satisfying:

$$A^*A = AA^*.$$

Let $A = [a_{ij}] \in M_n$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly equal, we do not assume here that A is diagonalizable). We are going to show that the following statements are equivalent:

(a) A is normal.

(b) A is unitarily diagonalizable (i.e. there exists U unitary such that U^*AU is diagonal).

(c) $\sum_{i,j=1}^n |a_{ij}|^2 = \sum_{i=1}^n |\lambda_i|^2$.

(d) A has n orthonormal eigenvectors.

Answer the following questions:

1. Show that *(b)* $\implies$ *(c)*.

2. Show that any diagonal matrix is normal. Show that any matrix unitarily similar to a normal matrix is also normal.

3. Show that *(d)* $\implies$ *(a)*

4. Let $A \in M_n$ be partitioned as
$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$
in which A_{11} and A_{22} are square. Show that A is normal if and only if A_{11} and A_{22} are normal and $A_{12} = 0$.

5. Show that *(a)* $\implies$ *(b)*.

6. Show that *(c)* $\implies$ *(d)*

[1] Recall that $P^{-*} = (P^*)^*$.

Problem 2 (25%): Sylvester equation.

Let us consider $A, B, C \in \mathcal{M}_n$.

1. Given a matrix $M \in \mathcal{M}_{m,n}$, define a vectorization procedure of M ($\text{Vec}(M) \in \mathbb{C}^{pn}$). Vectorize the equation $AX + XB = C$, $X \in \mathcal{M}_n$ and give the conidition for existence and uniqueness of the solution $\hat{X} \in \mathcal{M}_n$. Express $\text{Vec}(\hat{X})$ with the Kronecker product of A and B.

2. Show that $Z : t \mapsto e^{tA} C e^{tB}$ is solution to the differential equation:

$$\begin{cases} \dfrac{dZ}{dt} = AZ + ZB \\ Z(0) = C \end{cases}$$

3. Given $d \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ let us denote $J_d(\lambda)$ the Jordan block defined as:

$$J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \lambda \end{pmatrix} \in \mathcal{M}_d,$$

   express $e^{tJ_d(\lambda)}$.

4. Deduce that when A and B only have strictly negative eigenvalues, $\hat{X} = -\int_0^\infty e^{tA} C e^{tB} dt$ (we assume that this integral is well defined).

5. Show that if A is Hermitian, then e^A is also Hermitian. Show that if A is positive semi-definite then e^A is also positive semi definite.

6. Show that if $B = A^*$ and $-C$ is Hermitian positive semi-definite then X is also positive semidefinite.

Problem 3 (10%): Schur complement.

1. Let us consider a matrix $A_{11} \in \mathcal{M}_n$, $A_{12} \in \mathcal{M}_{n,p}$, $A_{12} \in \mathcal{M}_{p,n}$ and $A_{22} \in \mathcal{M}_{p,p}$. Assuming that A_{11} is invertible, compute the product

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix},$$

   and deduce that $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ is invertible if and only if its so called "Schur complement" $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is invertible.

2. Given $X \in M_{p,q}$ let us introduce

$$K = \begin{bmatrix} I_p & X \\ X^* & I_q \end{bmatrix} \in M_{p+q}.$$

   Show that K is positive definite if and only if X is a strict contraction (its singular values are all strictly lower than 1).

3. Given two positive semidefinite matrices $A, B \in \mathcal{M}_n$, show that the three following properties are equivalent:

   (a) $A \succ B$

   (b) $\rho(A^{-1}B) \leq 1$

   (c) There exists a contraction $X \in \mathcal{M}_n$ such that $B = A^{\frac{1}{2}} X A^{\frac{1}{2}}$.

4. Let $H = \left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right) \in M_{p+q}$ be Hermitian with $A \in M_p$ and $C \in M_q$. Show the equivalence:

   (a) H is positive definite.

   (b) A is positive definite and $C - B^* A^{-1} B$ is positive definite.

   (c) A and C are positive definite and $\rho(B^* A^{-1} B C^{-1}) < 1$.