Exercises

Exercise 1. *Suppose that $A \in M_n$ is diagonalizable.*

(a) Prove that the rank of A is equal to the number of its nonzero eigenvalues.

(b) Prove that rank $A =$ rank A^k for all $k = 1, 2, \ldots$.

(c) Prove that A is nilpotent if and only if $A = 0$.

(d) If tr$A = 0$, prove that rank $A \neq 1$.

(e) Use each of the four preceding results to show that $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is not diagonalizable.

Exercise 2. *(a) Let $A \in M_n$ and a polynomial $p(t)$ be given. If A is diagonalizable, show that $p(A)$ is diagonalizable. What about the converse?*

(b) If $A, B \in M_n$, and if A and B commute, show that any polynomial in A commutes with any polynomial in B.

Exercise 3. *Let $x, y \in \mathbb{C}^n$ be given and suppose that $y^*x \neq 1$.*

(a) Verify that $I + xy^*$ is invertible and $(I + xy^*)^{-1} = I - cxy^*$, in which $c = (1 + y^*x)^{-1}$.

(b) Let $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$ and suppose that $y^*x = 0$. Explain why the eigenvalues of $A = (I + xy^*)\Lambda(I - xy^*)$ are $\lambda_1, \ldots, \lambda_n$.

(c) Use this observation to construct a non-triangular 2-by-2 matrix with integer entries and eigenvalues $1, 2$.

Exercise 4. *(a) Let $a, b \in \mathbb{C}$. Show that the eigenvalues of*

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

are $a \pm ib$.

(b) Let $x \in \mathbb{C}^n$ be a given nonzero vector, and write $x = u + iv$, in which $u, v \in \mathbb{R}^n$. Show that the vectors $x, \bar{x} \in \mathbb{C}^n$ are linearly independent if and only if the vectors $u, v \in \mathbb{R}^n$ are linearly independent.

Exercise 5. *Show that a Hermitian $P \in M_n$ is a projection if and only if there is a unitary $U \in M_n$ such that $P = U(I_k \oplus 0_{n-k})U^*$, in which $0 \leq k \leq n$.*

Exercise 6. *Let $A, B \in M_n$ be given and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

(a) Show that $x^TAx = 0$ for all $x \in \mathbb{K}^n$ if and only if $A^T = -A$. Give an example to show that A and B need not be equal if $x^TAx = x^TBx$ for all $x \in \mathbb{K}^n$, so a real or complex matrix is not determined by the quadratic form that it generates.

(b) Show that $x^*Ax = x^*Bx$ for all $x \in \mathbb{C}^n$ if and only if $A = B$, that is, a complex matrix is determined by the sesquilinear form it generates.

Exercise 7. *Show that $A \in M_n$ is diagonalizable if and only if the following condition is satisfied for each eigenvalue λ of A: If $x \in \mathbb{C}^n$ and $(A - \lambda I)^2x = 0$, then $(A - \lambda I)x = 0$. (**Hint:** Use the Jordan decomposition of A)*

Exercise 8. *Suppose that $A \in M_n$ is nonzero. Show that*

$$rank\ A \geq \left|\frac{tr\ A}{2}\right|^2 / (tr\ A^* A)$$

with equality if and only if $A = aH$ for some nonzero $a \in \mathbb{C}$ and some Hermitian projection[1] *H.*

Exercise 9. *Given $A \in M_n$, show that $x^* A x$ is real and positive for all nonzero $x \in \mathbb{C}^n$ if and only if A is Hermitian and all of its eigenvalues are positive.*

Exercise 10. *(a) If$A, B \in M_n$, A is nonsingular, and B is singular, show that*

$$\|A - B\| \geq \frac{1}{\|A^{-1}\|}$$

(b) Can a nonsingular matrix be closely approximated by a singular matrix?

Exercise 11. *Consider the following robust least squares problem*

$$\min_{X \in \mathbb{R}^{m \times m}} \max_{\|\Delta\|_2 \leq \lambda} \left\|Y - (A + \Delta)X - X(A + \Delta)^T\right\|_F ,$$

where $\Delta \in \mathbb{R}^{m \times m}$ is an error matrix, $\lambda > 0$, $Y \in \mathbb{R}^{m \times m}$, and $A \in \mathbb{R}^{m \times m}$ are given.

(a) Show that the above problem has an upper bound problem given by

$$X^* = \arg \min_{X \in \mathbb{R}^{m \times m}} \left\|Y - AX - XA^T\right\|_F + 2\lambda \|X\|_F ,$$

(b) Find the the closed-form expression for X^*.

Exercise 12. *Let us consider $A, B, C \in \mathcal{M}_n$ such that A and B only have negative eigenvalues.*

(a) Solve the differential equation:

$$\begin{cases} \dfrac{dZ}{dt} = AZ + ZB \\ Z(0) = C \end{cases}$$

(b) Deduce the solution $X^* \in \mathcal{M}_n$ of the equation $AX + XB = C$, $X \in \mathcal{M}_n$ in the case where A and B only have strictly negative eigenvalues.

(c) Show that if $B = A^*$ and C is positive semi-definite then X is also positive semidefinite.

Exercise 13. *If $U \in M_n$ is unitary, show that $x, y \in \mathbb{C}^n$ are orthogonal if and only if Ux and Uy are orthogonal.*

Exercise 14. *A nonsingular matrix $A \in M_n$ is skew orthogonal if $A^{-1} = -A^T$.*

- *Show that A is skew orthogonal if and only if $\pm iA$ is orthogonal.*
- *More generally, if $\theta \in \mathbb{R}$, show that $A^{-1} = e^{i\theta} A^T$ if and only if $e^{i\theta/2} A$ is orthogonal. What is this for $\theta = 0$ and $\theta = \pi$?*

Exercise 15.

- *Show that the Frobenius norm and the matrix 2-norm are unitarily invariant, i.e., that*

$$\|PAQ\|_F = \|A\|_F \quad \text{and} \quad \|PAQ\|_2 = \|A\|_2$$

for all $A \in \mathbb{C}^{n,m}$ and unitary matrices $P \in \mathbb{C}^{n,n}$, $Q \in \mathbb{C}^{m,m}$.

- *Show that $\|A\|_F = (\sigma_1^2 + \ldots + \sigma_r^2)^{1/2}$ and $\|A\|_2 = \sigma_1$ where $\sigma_1 \geq \ldots \geq \sigma_r > 0$ are the singular values of $A \in \mathcal{M}_{n,m}$.*

[1] It means that H is Hermitian and satisfies $H^2 = H$.