Quizz 2

1. Given a subset $S \subset \mathbb{C}^n$, prove that $(S^\perp)^\perp = S$.
2. Given a polynomial $P \in \mathbb{C}[X]$ and a matrix $A \in \mathcal{M}_n(\mathbb{C})$ such that $P(A) = 0$, what can be said about $P(\lambda)$, for any $\lambda \in \mathbb{C}$, an eigenvalue of A. Justify the result.
3. Given matrix $A \in \mathcal{M}_n$, explain what is the characteristic polynomial χ_A and give the Cayley Hamilton Theorem.
4. Give the definition of algebraic and geometric multiplicity of an eigenvalue and express the Jordan decomposition of a matrix that has two eigenvalues 0 and 3 with respective algebraic multiplicity $3, 2$ and respective geometric multiplicity $2, 1$. ***(Hint:** The algebraic multiplicity is linked to the characteristic polynomial)*
5. Give Schur result about the triangularizibility of matrices, and explain what happens for the triangularizibility in $\mathcal{M}_n(\mathbb{C})$.
6. Prove the previous result on the Schur decomposition.
7. Give the Cauchy product formula for matrices and apply it when possible to express the exponential of a sum of matrices.
8. Give the definition of matrix norm and show that for any matrix norm $\|\cdot\|$ on $\mathcal{M}_n$, and any invertible matrix $P \in \mathcal{M}_n$ the mapping $M \mapsto \|P^{-1}MP\|$ is also a matrix norm.
9. Give the definitions of the Spectral Radius and of nilpotent matrices and express the spectral radius of a nilpotent matrix; justify.
10. Show that the eigenvalues of a Hermitian matrix are real and that two eigenvectors associated to two distinct eigenvalues are orthogonal.
11. Show that a Hermitian matrix is diagonalizable with a unitary matrix.
12. Give Weyl theorem on the control of the eigenvalues of a sum of matrices.
13. Show thanks to Weyl Theorem that given two Hermitian matrices $A, B \in \mathcal{H}_n$:
$$A \succeq B \quad \implies \quad \forall i \in [n]: \; \lambda_i(A) \geq \lambda_i(B),$$
where $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ (resp. $\lambda_1(B) \leq \cdots \leq \lambda_n(B)$) are the ordered list of eigenvalues of A (resp. B).
14. Give the definition of projections, and reflexions and show that reflection matrices are involutions (they are their own inverses).
15. Give the QR Algorithm to find eigenvalues.
16. Define the Moore Penrose inverse.
17. Provide a matrix of $\mathcal{M}_2$ that do not admit a LU decomposition and justify.
18. Provide a matrix of $\mathcal{M}_2$ that is not diagonalizable and justify.